Question 10

A closed surface with Euler characteristic χ is to be given a regular subdivision with F faces, V vertices and E edges. Each face of the subdivision is to be a j-gon, with 5 faces meeting at each vertex.

(i) Write down a formula connecting F, j and V. [1]

(ii) Write down a formula connecting F, j and E. [1]

(iii) Write down a formula connecting F, j and χ. [1]

(iv) Hence find all possible values for F, V and E in regular subdivisions of this type when

(a) $\chi = 2$;

(b) $\chi = 1$;

(c) $\chi = 0$;

(d) $\chi = -1$. [8]

Question 11

(i) Reduce the edge equation $abcab^{-1} = 1$ to canonical form. Hence or otherwise classify the surface it defines as one of the following:

$$D^2,\ 2D^2,\ \mathbb{RP}^2,\ \mathbb{RP}^2\#D^2,\ 2\mathbb{RP}^2,\ 2\mathbb{RP}^2\#D^2.$$ [4]

(ii) Reduce the edge equation $ab^{-1}cd^{-1}a^{-1}bc^{-1}d = 1$ to canonical form. Hence or otherwise write the surface it defines as one of the following:

$$T^2,\ \mathbb{RP}^2,\ 2T^2,\ \mathbb{RP}^2\#D^2,\ 2\mathbb{RP}^2,\ 3\mathbb{RP}^2,\ 4\mathbb{RP}^2.$$ [3]

(iii) Reduce the simultaneous edge equations

$$abac = 1, \quad cded = 1, \quad efgf^{-1} = 1$$

to canonical form. Hence or otherwise write the surface it defines as one of the following:

$$\mathbb{RP}^2,\ \mathbb{RP}^2\#D^2,\ \mathbb{RP}^2\#2D^2,\ 2\mathbb{RP}^2\#D^2,\ 2\mathbb{RP}^2\#2D^2.$$ [4]

Question 12

We shall say that two surfaces M and N are equivalent (and write $M \sim N$) if and only if there exists an integer p such that $M\#p\mathbb{RP}^2 = N\#p\mathbb{RP}^2$. You may assume without proof that $\sim$ is an equivalence relation.

(i) Show that $T^2 \sim 2\mathbb{RP}^2$ [3]

(ii) Show that if m and d are integers with $m \geq 1$, then

$$mT^2\#dD^2 \sim 2m\mathbb{RP}^2\#dD^2.$$ [3]

(iii) Show that the only surface equivalent to the sphere S^2 is S^2 itself.

[**Hint**: assume $X \sim S^2$, where $X = aT^2\#b\mathbb{RP}^2\#dD^2$.] [3]

(iv) Show that the only surface equivalent to $\mathbb{RP}^2$ is $\mathbb{RP}^2$ itself. [2]