Question 7

(i) Prove that if H is a connected subspace of a topological space T and $H \subset K \subset Cl(H)$ then K is connected. [5]

(ii) For each positive integer n, let

$$L_n = \{(x_1, x_2) \in \mathbf{R}^2 : x_2 = nx_1,\ x_1^2 + x_2^2 < 1\},$$

let

$$J = \left\{(x_1, x_2) \in \mathbf{R}^2 : x_1 = 0,\ \frac{1}{3} < |x_2| < \frac{2}{3}\right\}$$

and let

$$K = J \cup \bigcup_{n=1}^{\infty} L_n.$$

Prove that the subspace K of $\mathbf{R}^2$ with the usual topology is connected. [6]

Question 8

Let d be the metric on the set $\mathbf{N}$ of positive integers defined by

$$d(i, j) = \left|\frac{2i}{i+1} - \frac{2j}{j+1}\right|, \quad i, j \in \mathbf{N}.$$

[You need *not* verify that d is a metric.]

Put $M = (\mathbf{N}, d)$ and let (x_n) be the sequence in the metric space M given by $x_n = n$ for each positive integer n.

(i) Show that if $m > n$, then $d(x_n, x_m) < \frac{2}{n+1}$. Hence or otherwise prove that (x_n) is a Cauchy sequence in the metric space M. [4]

(ii) Show that if $k \in \mathbf{N}$ then

$$d(k, x_n) \geq \frac{1}{k+1}$$

for each integer n such that $n \geq 2k + 1$. [3]

(iii) Deduce that the metric space M is not complete. [4]