Question 4

Let $T_1 = \{A, \mathcal{T}\}$ where $A = \{a, b, c\}$ and $\mathcal{T}$ is the topology

$$\{\varnothing, \{a\}, \{a, b\}, \{a, c\}, A\}$$

and let $T_2 = \{\mathbf{Z}, \mathcal{V}\}$ where $\mathbf{Z}$ is the set of integers and $\mathcal{V}$ is the topology defined as follows: $V \in \mathcal{V}$ if either $0 \notin V$ or $1 \in V$.

[You need *not* check that $\mathcal{T}$, $\mathcal{V}$ are topologies.]

(i) Prove that $f : A \longrightarrow \mathbf{Z}$ given by

$$f(a) = f(b) = 1, \quad f(c) = 0$$

is $(\mathcal{T}, \mathcal{V})$-continuous. [4]

(ii) Prove that $g : A \longrightarrow \mathbf{Z}$ given by

$$g(a) = -1, \quad g(b) = g(c) = 1$$

is not $(\mathcal{T}, \mathcal{V})$-continuous. [3]

(iii) State whether or not $h : A \longrightarrow \mathbf{Z}$ given by

$$h(a) = 1, \quad h(b) = h(c) = 0$$

is $(\mathcal{T}, \mathcal{V})$-continuous, justifying your answer. [4]

Question 5

Let A denote the interval $(0, 1]$ in $\mathbf{R}$ and let $\mathcal{T}$ be the topology on A which consists of $\varnothing$, A and all intervals of the form $(a, 1]$ where $a \in A$.

[You need *not* verify that $\mathcal{T}$ is a topology on A.]

Put $T = \{A, \mathcal{T}\}$ and let $H = \{\frac{1}{3}, \frac{1}{2}\}$.

(i) Prove that if $x \in A$ and $x < \frac{1}{2}$ then x is a limit point of H in T. [3]

(ii) Prove that if $x \in A$ and $x \geq \frac{1}{2}$ then x is not a limit point of H in T. [5]

(iii) Determine the closure of H in T. [3]

Question 6

(i) Prove that a compact subspace of a metric space is bounded. [5]

(ii) For each positive integer n, let

$$A_n = \left\{ (x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 = \frac{1}{n^2} \right\}$$

and let

$$A = \bigcup_{n=1}^{\infty} A_n,$$
$$B = A \cup \{(0, 0)\},$$
$$C = A \cup \{(x_1, x_2) \in \mathbf{R}^2 : x_1 = 0\}.$$

For each of the subsets A, B and C of $\mathbf{R}^2$ with the usual topology, state whether or not it is compact, briefly justifying your answer. [6]