Answer as many questions as you wish. Full marks may be obtained by complete *answers to* NINE *questions, provided that no more than* SIX *questions have been selected from any one part. All questions carry equal marks.*

PART I METRIC AND TOPOLOGICAL SPACES

Question 1

(i) Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3, 4\}$ and let $f : A \longrightarrow B$ be given by

$$f(a) = f(b) = 1, \quad f(c) = 2, \quad f(d) = 3.$$

Write down:

(a) $f(f^{-1}(\{1, 2, 3\}))$;

(b) $f(f^{-1}(\{2, 3, 4\}))$. [4]

(ii) Let $f : C \longrightarrow D$ be a function, where C and D are non-empty sets, and let E be a subset of D.

(a) Prove that

$$f(f^{-1}(E)) \subset E.$$ [3]

(b) Prove that if $E \subset f(C)$ then

$$E = f(f^{-1}(E)).$$ [4]

Question 2

Let $d : \mathbf{R} \times \mathbf{R} \longrightarrow \mathbf{R}$ be defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ |x| + |y| + 2|x - y|, & \text{if } x \neq y. \end{cases}$$

(i) Show that d is a metric on $\mathbf{R}$. [5]

(ii) Determine the following open balls with respect to the metric d:

(a) $B_\delta(0)$, where $\delta > 0$;

(b) $B_1(1)$. [3]

(iii) Show that the function $f : \mathbf{R} \longrightarrow \mathbf{R}$ defined by

$$f(x) = 2x + 1, \quad x \in \mathbf{R},$$

is not (d, d)-continuous at 0. [3]

Question 3

Let $\mathscr{T}$ be the collection of subsets of the set $\mathbf{Z}$ of integers defined as follows: $V \in \mathscr{T}$ if

either $\{-1, 1\} \cap V = \varnothing$
or $E \subset V$ where E is the set of all even integers.

(i) Show that $\mathscr{T}$ is a topology on $\mathbf{Z}$. [7]

(ii) Determine the topology induced by $\mathscr{T}$ on the set $\{-1, 0, 1, 2\}$. [4]