Question 7

(i) Prove that if H is a connected subspace of a topological space T and $H \subset K \subset Cl(H)$, then K is connected. [5]

(ii) For each positive integer n, let L_n be the line segment in $\mathbb{R}^2$ joining $(0,0)$ to $\left(\frac{1}{n}, 1\right)$, let

$$J = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0,\ 0 \le x_2 < \frac{1}{4} \text{ or } \frac{3}{4} < x_2 \le 1\right\}$$

and let

$$K = J \cup \bigcup_{n=1}^{\infty} L_n.$$

Prove that the subspace K of $\mathbb{R}^2$ with the usual topology is connected. [6]

Question 8

(i) Prove that a convergent sequence in a metric space M is a Cauchy sequence in M. [4]

(ii) Let $\mathbb{N}$ denote the set of positive integers and let d be the metric on $\mathbb{N} \times \mathbb{N}$ defined by

$$d((i_1, j_1), (i_2, j_2)) = \left|\frac{1}{i_1} - \frac{1}{i_2}\right| + |j_1 - j_2|$$

for $(i_1, j_1), (i_2, j_2) \in \mathbb{N} \times \mathbb{N}$.
[You need not verify that d is a metric.]

(a) Prove that if $x_n = (2n, 1)$ for each positive integer n, then (x_n) is a Cauchy sequence in the metric space $(\mathbb{N} \times \mathbb{N}, d)$. [4]

(b) Prove that if $y_n = (1, n)$ for each positive integer n, then (y_n) is *not* a Cauchy sequence in the metric space $(\mathbb{N} \times \mathbb{N}, d)$. [3]