Answer as many questions as you wish. Full marks may be obtained by complete answers to NINE questions, provided that no more than SIX questions have been selected from any one part. All questions carry equal marks.

PART I METRIC AND TOPOLOGICAL SPACES

Question 1

(i) Let $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$ and let $f : A \longrightarrow B$ be given by

$$f(a) = f(b) = 1, \quad f(c) = 2, \quad f(d) = 3.$$

Write down:

(a) $f^{-1}(f(\{b, c\}))$;

(b) $f^{-1}(f(\{c, d\}))$. [4]

(ii) Let $f : G \longrightarrow H$ be a function, where G and H are non-empty sets.

(a) Prove that if $E \subset G$, then

$$E \subset f^{-1}(f(E)).$$ [3]

(b) Prove that if f is injective and $E \subset G$ then

$$f^{-1}(f(E)) = E.$$ [4]

Question 2

Let d: $\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ |x| + |y| + |x - y|, & \text{if } x \neq y. \end{cases}$$

(i) Show that d is a metric on $\mathbb{R}$. [5]

(ii) Determine the following open balls with respect to the metric d:

(a) $B_1(0)$;

(b) $B_1(1)$. [3]

(iii) Show that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = 1 + x, \quad x \in \mathbb{R},$$

is not (d, d)-continuous at 0. [3]

Question 3

Let $\mathcal{T}$ be the collection of subsets of the set $\mathbb{N}$ of positive integers defined as follows: $V \in \mathcal{T}$ if *either* $1 \notin V$ *or* $\mathbb{O} \subset V$ where $\mathbb{O}$ is the set of odd positive integers.

(i) Show that $\mathcal{T}$ is a topology on $\mathbb{N}$. [7]

(ii) Determine the topology induced by $\mathcal{T}$ on the set $\{1, 2, 3\}$. [4]