Question 4

Let $T_1 = \{A, \mathcal{T}\}$ where $A = \{a, b, c\}$ and $\mathcal{T}$ is the topology

$$\Big\{\emptyset, \{a\}, \{a, b\}, \{a, c\}, A\Big\}$$

and let $T_2 = \{\mathbb{Z}, \mathcal{V}\}$ where $\mathbb{Z}$ is the set of all integers and $\mathcal{V}$ is the topology defined as follows: $V \in \mathcal{V}$ if *either* $0 \notin V$ *or* $1 \in V$.

[You need not check that $\mathcal{T}$, $\mathcal{V}$ are topologies.]

(i) Prove that $f : A \longrightarrow \mathbb{Z}$ given by

$$f(a) = f(b) = 1, \quad f(c) = 0$$

is $(\mathcal{T}, \mathcal{V})$-continuous. [4]

(ii) Prove that $g : A \longrightarrow \mathbb{Z}$ given by

$$g(a) = 0, \quad g(b) = g(c) = 1$$

is not $(\mathcal{T}, \mathcal{V})$-continuous. [3]

(iii) State whether or not $h : A \longrightarrow \mathbb{Z}$ given by

$$h(a) = -1, \quad h(b) = h(c) = 0$$

is $(\mathcal{T}, \mathcal{V})$-continuous, justifying your answer. [4]

Question 5

Let $\mathbb{N}$ denote the set of positive integers and let $\mathcal{B} = (B_n)_{n\in\mathbb{N}}$ be the collection of subsets of $\mathbb{N}$ defined as follows:

$$B_n = \begin{cases} \{n\}, & \text{if } n \text{ is even,} \\ \{n\} \cup \{m : m \text{ even and } m > n\}, & \text{if } n \text{ is odd.} \end{cases}$$

(i) Prove that $\mathcal{B}$ is a synthetic basis for $\mathbb{N}$. [5]

(ii) Prove that the set $\mathbb{O}$ of odd positive integers is closed in $\{\mathbb{N}, \mathcal{T}\}$, where $\mathcal{T}$ is the topology on $\mathbb{N}$ arising from $\mathcal{B}$. [2]

(iii) (a) Show that 3 is a limit point of $\{2, 4\}$ in $\{\mathbb{N}, \mathcal{T}\}$. [2]

(b) Write down the closure of $\{2, 4\}$ in $\{\mathbb{N}, \mathcal{T}\}$. [2]

Question 6

(i) Prove that each compact subspace of a metric space M is bounded in M. [5]

(ii) For each of the following subsets of $\mathbb{R}^2$ with the usual topology, state whether or not it is compact, justifying your answer:

$H_1 = \{(x_1, x_2) : x_1^2 \le x_2\}$; [2]

$H_2 = \{(x_1, x_2) : x_1^2 \le x_2 < 1\}$; [2]

$H_3 = \{(x_1, x_2) : x_1^2 \le x_2 \le 1\}$. [2]