Question 4

Let $\mathcal{T}$ be the topology on the set $\mathbf{Z}$ of integers consisting of $\mathbf{Z}$ and all subsets of $\mathbf{Z}$ which do not contain 0 and let $\mathcal{U}$ be the topology on $\mathbf{Z}$ consisting of $\emptyset$ and all subsets of $\mathbf{Z}$ which do contain 0.
[You need **not** verify that $\mathcal{T}$ and $\mathcal{U}$ are topologies.]

(i) Prove that $f : \mathbf{Z} \longrightarrow \mathbf{Z}$ given by

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x \neq 0 \end{cases}$$

is neither $(\mathcal{T}, \mathcal{U})$ – continuous nor $(\mathcal{U}, \mathcal{T})$ – continuous. [5]

(ii) Prove that $g : \mathbf{Z} \longrightarrow \mathbf{Z}$ given by

$$g(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

is both $(\mathcal{T}, \mathcal{U})$ – continuous and $(\mathcal{U}, \mathcal{T})$ – continuous. [6]

Question 5

Let $\mathcal{T}$ be the topology on $\mathbf{R}$ which consists of all sets V such that either $1 \notin V$ or there exists a positive real number r such that $(1 - r,\ 1 + r) \subset V$, and put $T = \{\mathbf{R}, \mathcal{T}\}$.
[You need **not** verify that $\mathcal{T}$ is a topology on $\mathbf{R}$.]

(i) Describe the closed sets of the topological space T. [3]

(ii) For each of the following subsets of $\mathbf{R}$, state whether or not it is a closed set of T and justify your answer:

(a) $(0, 2)$;

(b) $\{0, 2\}$;

(c) $\{0, \frac{1}{2}, \frac{2}{3}, \ldots, \frac{n}{n+1}, \ldots\}$. [6]

(iii) For any set in part (ii) which is not closed in T, write down its closure in T. [2]

Question 6

(i) Prove that a compact subspace of a metric space is bounded. [5]

(ii) For each of the following subsets of $\mathbf{R}^2$ with the usual topology, state whether or not it is compact, justifying your answer.

$$H_1 = \{(x_1, x_2) : x_1^2 + x_2^2 \geq 1\}$$
$$H_2 = \{(x_1, x_2) : x_1^2 + x_2^2 \geq 1, |x_1| + |x_2| < 2\}$$
$$H_3 = \{(x_1, x_2) : x_1^2 + x_2^2 \geq 1, \operatorname{Max}\{|x_1|, |x_2|\} \leq 1\}$$ [6]

Question 7

(i) Prove that if A_1 and A_2 are connected subspaces of a topological space such that $A_1 \cap A_2 \neq \emptyset$, then $A_1 \cup A_2$ is connected. [5]

(ii) For each positive integer n, let J_n be the line-segment in $\mathbf{R}^2$ joining $(0, 0)$ to $(1, \frac{1}{n})$ and let

$$K = \{(x, 0) \in \mathbf{R}^2 : x \geq 1\}.$$

Prove that the subspace

$$\left(\bigcup_{n=1}^{\infty} J_n\right) \cup K$$

of $\mathbf{R}^2$ with the usual topology is connected. [6]