Answer as many questions as you wish. Full marks may be obtained by **complete** *answers to* NINE *questions, provided that no more than* SIX *questions have been selected from any one part. All questions carry equal marks.*

PART I METRIC AND TOPOLOGICAL SPACES

Question 1

(i) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function given by

$$f(x_1, x_2) = 1 + x_1^2 + x_2^2 \quad , \quad (x_1, x_2) \in \mathbb{R}^2.$$

(a) Sketch the subset

$$f^{-1}([-1, 2])$$

of $\mathbb{R}^2$. [3]

(b) Write down $f(f^{-1}([-1, 2]))$. [2]

(ii) Let $g : A \longrightarrow B$ be a function, where A and B are non-empty sets.

(a) Prove that if $K \subset B$ then

$$g(g^{-1}(K)) \subset K.$$ [3]

(b) Prove that if g is onto and $K \subset B$ then

$$g(g^{-1}(K)) = K.$$ [3]

Question 2

Let $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 2|x| + 2|y| + |x - y| & \text{if } x \neq y. \end{cases}$$

(i) Show that d is a metric on $\mathbb{R}$. [5]

(ii) Determine the following open balls with respect to the metric d:

(a) $B_1(0)$; (b) $B_1(1)$. [3]

(iii) Show that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = 1 + x \quad , \; x \in \mathbb{R},$$

is not (d, d)–continuous at 0. [3]

Question 3

Let $A = \{0, 1, 2, \ldots\}$ be the set of non-negative integers and let $\mathcal{V}$ be the set of subsets of A defined as follows: $V \in \mathcal{V}$ if $0 \notin V$ or $E \subset V$ where E is the set of even positive integers.

(i) Show that $\mathcal{V}$ is a topology on A. [7]

(ii) Determine the topology induced by $\mathcal{V}$ on the set $\{0, 1, 2\}$. [4]