Question 8

Let $\mathbf{N}$ denote the set of positive integers and let d be the metric on $\mathbf{N} \times \mathbf{N}$ defined by

$$d((i,j),(i_1,j_1)) = |i - i_1| + \left|\frac{1}{j} - \frac{1}{j_1}\right|$$

for $(i,j), (i_1,j_1) \in \mathbf{N} \times \mathbf{N}$.
[You need **not** verify that d is a metric.]
Put $M = (\mathbf{N} \times \mathbf{N}, d)$ and let (x_n) be the sequence in the metric space M given by $x_n = (1,n)$ for each positive integer n.

(i) Prove that (x_n) is a Cauchy sequence in M. [3]

(ii) Show that if $y = (i,k)$ where $i \neq 1$, then (x_n) does not converge to y in M. [2]

(iii) Show that if $k \in \mathbf{N}$ then

$$d((1,k),x_n) \geq \frac{1}{2k}$$

for each integer n such that $n \geq 2k$. [3]

(iv) Deduce that the metric space M is not complete. [3]