Question 8

(i) Prove that a complete subspace of a metric space M is closed in M. [4]

(ii) Let d be the metric on the set $\mathbb{N}$ of positive integers defined by

$$d(i,j) = \left|\frac{1}{i} - \frac{1}{j}\right|, \qquad i, j \in \mathbb{N}.$$

(a) Prove that if $x_n = 2^n$, $n \in \mathbb{N}$, then (x_n) is a Cauchy sequence in the metric space $\{\mathbb{N}, d\}$. [3]

(b) Show that if $k \in \mathbb{N}$, then

$$d(k, x_n) > \frac{1}{2k},$$

for each integer n such that $n \geqslant 2k$. [2]

(c) Deduce that the metric space $\{\mathbb{N}, d\}$ is not complete. [2]