THE OPEN UNIVERSITY

Mathematics: A Third Level Course

M386 Metric and Topological Spaces and Geometric Topology

M386 Solutions to Specimen Examination Paper

Part I Metric and Topological Spaces

Question 1

(i) (a) $f^{-1}(\{1,2\}) = \{a,b\}$ so that $f(f^{-1}\{1,2\})) = \{2\}$.

(b) $f^{-1}(\{2,3\}) = \{a,b,c\}$ so that $f(f^{-1}(\{2,3\})) = \{2,3\}$.

(ii) (a) If $y \in f(f^{-1}(D))$, then $y = f(x)$ where $x \in f^{-1}(D)$. But since $x \in f^{-1}(D)$ it follows that $y = f(x) \in D$. Hence

$$f(f^{-1}(D)) \subset D.$$

(b) If $y \in D$ then since f is onto there exists $x \in A$ such that $f(x) = y$. Since $f(x) \in D$ it follows that $x \in f^{-1}(D)$.Thus $y = f(x)$ where $x \in f^{-1}(D)$ and so $y \in f(f^{-1}(D))$. Thus $D \subset f(f^{-1}(D))$ so that from (a) we have

$$D = f(f^{-1}(D)).$$

Question 2

(i) We must check that (M1), (M2) and (M3) are satisfied.

M1 If $x = y$, then $d(x,y) = 0$.
If $x \neq y$, then either $x \neq 1$ or $y \neq 1$ so $d(x,y) > 0$.

M2 is obviously satisfied.

M3 We need only verify that the triangle inequality holds for x, y, z all different.

$$\begin{aligned} d(x,y) + d(y,z) &= |x-1| + 2|y-1| + |z-1| \\ &\geqslant |x-1| + |z-1| = d(x,z), \end{aligned}$$

since $|y-1| \geqslant 0$.

Since d satisfies (M1), (M2) and (M3), it is a metric on R.

(ii) (a) If $y \neq 0$, then $d(0,y) = 1 + |y-1|$. There is no y such that $1 + |y-1| < 1$ and so $B_1(0) = \{0\}$.

(b) Since $d(1,y) = |y-1|$, we see that $B_1(1) = (0,2)$.

(iii) Suppose f is (d,d)-continuous at 1. Then there exists $\delta > 0$ such that $f(x) \in B_1(0)$ if $x \in B_\delta(1)$. Thus there exists $\delta > 0$ such that $f(x) = 0$ if $x \in (1-\delta, 1+\delta)$. Since $f(1-\frac{1}{2}\delta) = \frac{1}{2}\delta \neq 0$, we have a contradiction. Hence f is not (d,d)-continuous at 1.

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