Question 3

(i) We must show that (T1), (T2) and (T3) are satisfied.

T1 Since $1 \notin \emptyset$, $\emptyset \in \mathcal{T}$.
Since $E \subset \mathbf{N}$, $\mathbf{N} \in \mathcal{T}$.

T2 Let $V_1, V_2 \in \mathcal{T}$. If $1 \notin V_1 \cap V_2$ then $V_1 \cap V_2 \in \mathcal{T}$. If $1 \in V_1 \cap V_2$ then $1 \in V_1$ and $1 \in V_2$. Since $V_1 \in \mathcal{T}$ and $1 \in V_1$ it follows that $E \subset V_1$. Similarly $E \subset V_2$. Thus $E \subset V_1 \cap V_2$ and so $V_1 \cap V_2 \in \mathcal{T}$.

T3 Let $(V_i)_{i \in I}$ be a family of elements of $\mathcal{T}$ and let $V = \bigcup_{i \in I} V_i$. If $1 \notin V$ then $V \in \mathcal{T}$. If $1 \in V$ then there exists $j \in I$ such that $1 \in V_j$. Since $V_j \in \mathcal{T}$ and $1 \in V_j$ it follows that $E \subset V_j$. Hence $E \subset V_j \subset V$ and so $V \in \mathcal{T}$.

(ii) (a) Put $H = \{n \in \mathbf{N} : n \text{ is odd}\}$. If $A \subset H$, then $A \cup E \in \mathcal{T}$. Since $(A \cup E) \cap H = A$, we see that A belongs to the topology induced on H by $\mathcal{T}$. Thus the topology induced on H by $\mathcal{T}$ is the discrete topology.

(b) Put $K = \{1, 2\}$ and let $\mathcal{T}_1$ be the topology induced on K by $\mathcal{T}$. Suppose $W \in \mathcal{T}_1$ and $1 \in W$. Then there exists $V \in \mathcal{T}$ such that $V \cap K = W$. Since $1 \in V$, it follows that $E \subset V$ and so $W = V \cap K = K$. Since $\{2\} = \{2\} \cap K$ and $\{2\} \in \mathcal{T}$ we see that

$$\mathcal{T}_1 = \{\emptyset, \{2\}, K\}.$$

Question 4

(i) Let V be open in $\mathcal{T}_2$. Then there is a collection $(B_i)_{i \in I}$ of members of $\mathcal{B}$ much that $V = \bigcup_{i \in I} B_i$. Thus

$$f^{-1}(V) = f^{-1}\left(\bigcup_{i \in I} B_i\right) = \bigcup_{i \in I} f^{-1}(B_i).$$

Since $f^{-1}(B_i)$ is open in $\mathcal{T}_1$ for each i, it follows that $f^{-1}(V)$ is open in $\mathcal{T}_1$.

(ii) Suppose first that $g(b) \neq g(c)$ and put $W = \mathbf{R} - \{g(b)\}$. Then W is open in $\mathcal{T}_2$ and since $g(c) \in W$ and $g(b) \notin W$, it follows that $c \in g^{-1}(W)$ and $b \notin g^{-1}(W)$. But the open sets of $\mathcal{T}_1$ which contain c are $\{b, c\}$ and A. Thus $g^{-1}(W)$ is not open in $\mathcal{T}_1$. Hence $g : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ is not continuous.
Now suppose that $g(b) = g(c) = t$, say. If V is open in $\mathcal{T}_2$, then

$$g^{-1}(V) = \begin{cases} \emptyset \text{ or } \{a\} & \text{if } t \notin V \\ \{b, c\} \text{ or } A & \text{if } t \in V \end{cases}$$

Hence $g^{-1}(V)$ is open in $\mathcal{T}_1$. Therefore $g : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$ is continuous.

Question 5

(i) Since $\mathbf{N} \in \mathcal{B}$, (B1) is satisfied.
If $m, n \in \mathbf{N}$ and $m < n$ then $B_m \cap B_n = B_m$. Thus (B2) is satisfied.
Since (B1) and (B2) are satisfied, $\mathcal{B}$ is a synthetic basis for $\mathbf{N}$.

(ii) $\bigcup_{m=1}^{\infty} B_m$ is open in $\{\mathbf{N}, \mathcal{T}\}$. But $\bigcup_{m=1}^{\infty} B_m$ is the set of odd positive integers. Thus

$$E = \mathbf{N} - \bigcup_{m=1}^{\infty} B_m$$

is closed in $\{\mathbf{N}, \mathcal{T}\}$.