(iii) Put $H = \{2, 6, 7, 8\}$. If $k \in \mathbf{N}$ and k is even, then $\mathbf{N}$ is only open set which contains k and so k is a limit point of H. If $m > 4$ then every open set which contains $2m-1$ contains B_m; since $B_m \cap H = \{7\} \neq \emptyset$, $2m-1$ is a limit point of H. Thus

$$Cl(H) = (H) \cup \{n \in \mathbf{N} : n = 4 \text{ or } n \geqslant 9\}.$$

Question 6

(i) S Proposition 5.4.2, page 82.

(ii) (a) Not compact, as the subspace is not bounded.

(b) Compact, as the subspace is closed and bounded.

(c) Not compact, as the subspace is not closed.

Question 7

(i) S Proposition 6.4.1, page 102.

(ii) The subspace $A = H \cup \bigcup_{n=1}^{\infty} L_n$ is path-connected and hence connected. Let $\varepsilon > 0$ be given, and choose N such that $\frac{1}{N} < \varepsilon$.

Then $\left(\frac{1}{N}, 1\right) \in B_\varepsilon((0,1)) \cap L_N$. Thus $(0,1) \in Cl(A)$. Since $A \subset K \subset Cl(A)$ and A is connected, it follows from S Proposition 6.2.18, page 99, that K is connected.

Question 8

(i) S Proposition 9.2.3(a), page 123.

(ii) (a) If $m, n \in \mathbf{N}$ and $m < n$ then

$$d(x_n, x_m) = \left|\frac{1}{2^n} - \frac{1}{2^m}\right| = \frac{1}{2^m} - \frac{1}{2^n} < \frac{1}{2^m}.$$

Let $\varepsilon > 0$ be given. There exists a positive integer N such that $\frac{1}{2^N} < \varepsilon$. If $m, n \in \mathbf{N}$ and $n > m \geqslant N$ then

$$d(x_n, x_m) < \frac{1}{2^m} \leqslant \frac{1}{2^N} < \varepsilon.$$

Thus (x_n) is a Cauchy sequence in $\{\mathbf{N}, d\}$.

(b) If $n \geqslant 2k$ then

$$d(k, x_n) = \left|\frac{1}{k} - \frac{1}{2^n}\right| = \frac{1}{k} - \frac{1}{2^n} \geqslant \frac{1}{k} - \frac{1}{2^{2k}} > \frac{1}{k} - \frac{1}{2k} = \frac{1}{2k}.$$

(c) Suppose that $\{\mathbf{N}, d\}$ is complete. Then the Cauchy sequence (x_n) is convergent. Suppose that (x_n) converges to $k \in \mathbf{N}$. Then there exists a positive integer N such that $d(k, x_n) < \frac{1}{2k}$ for all $n \geq N$. But there exists $n \in \mathbf{N}$ such that $n \geqslant N$ and $n \geqslant 2k$. We then have a contradiction of (b). Therefore $\{\mathbf{N}, d\}$ is not complete.