Question 12

Determine all the distinct surfaces of Euler characteristic

(i) 2,

(ii) 1,

(iii) 0,

(iv) -1,

using connected sum notation.

For each surface, write down the number of its boundary curves and state whether or not it is orientable. [11]

Question 13

Categorize each of the following conics (which you may assume are non-degenerate) as one of the following:

(a) homeomorphic to the z-sphere;
(b) a double cover of the z-sphere branched over two points;
(c) a double cover of the z-sphere branched over two points and having a pinchpoint.

In each case locate the branch points and pinch points, where they exist.

(i) $z^2 + 2zw + 6 = 0$ [3]

(ii) $z^2 + 2zw + 4w^2 + 6 = 0$ [4]

(iii) $z^2 + 2zw + w^2 + 6z = 0$ [4]

Question 14

(i) Write down the Euler characteristic of an orientable surface of genus g with b boundary components. [2]

(ii) Let S be an orientable surface of genus g with 2 boundary curves, and let $\tilde{S}$ be an orientable surface of genus $\tilde{g} = g + k$ with $\tilde{b}$ boundary curves. Suppose $\tilde{S}$ is an unbranched n-fold cover of $S (n > 1)$. Using the inequalities relating $\tilde{b}$ to b and nb, show that

$$(n-1)(g-1) \leqslant k \leqslant (n-1)g.$$ [6]

(iii) Deduce that if $n = 3$ and $g > 2$, then $\tilde{g}$ cannot be $2g$. [3]

Question 15

(i) Show that the zeros of $F(z) = z^4 + 2z + 5$ all lie in the annulus $1 \leqslant |z| \leqslant 2$. [5]

(ii) Show that F has no real or imaginary zeros, and that its zeros are distributed one in each quadrant. [6]