Question 15

(i) On $C_1 = \{z \in \mathbb{C} : |z| = 1\}$, set $f(z) = z^4 + 2z$, $g(z) = 5$.
Then $|f(z)| = |z^4 + 2z| \leqslant |z|^4 + 2|z| = 3 < 5 = |g(z)|$.
Hence $V_{C_1}(f+g) = V_{C_1}(g) = 0$ and there are no zeros of F inside C_1 (Rouché's Theorem).
On $C_2 = \{z \in \mathbb{C} : |z| = 2\}$, set $f(z) = 2z + 5$, $g(z) = z^4$.
Then $|f(z)| = |2z + 5| \leqslant 2|z| + 5 = 9 < 16 = |g(z)|$.
Hence $V_{C_2}(f+g) = V_{C_2}(g) = 8\pi$ and so F has 4 zeros within C_2 (by Rouché's Theorem).
This proves part (i).

(ii) Consider $F(x)$, $x \in \mathbb{R}$: $F(x) = x^4 + 2x + 5$.
$F'(x) = 4x^3 + 2 = 0$ when $x = \sqrt[3]{-\frac{1}{2}}$, giving an obvious minimum.
Then $F(\sqrt[3]{-\frac{1}{2}})$ is roughly $(-0.8)^4 + 2(-0.8) + 5 > 0$.
so F is never zero on the real axis.
Next consider $z = iy$, in which case $F(z) = y^4 + 2iy + 5$.
This cannot be zero, since when $iy = 0$, $y^4 + 5 = 5$.
Now consider $C_3 = \{z = iy : -2 \leqslant y \leqslant 2\}$
and $C_4 = \{z \in C_2 : \mathrm{Re}(z) \geqslant 0\}$.

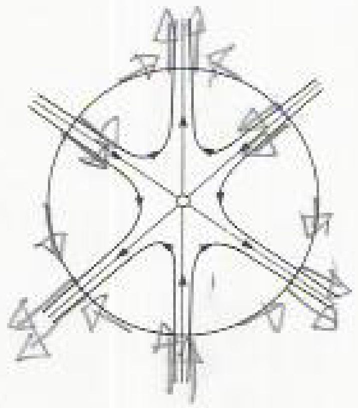
$V_{C_3}(F) = V_{C_3}(y^4 + 2iy + 5) < \pi$
$V_{C_4}(F) = 4\pi + \varepsilon$, $|\varepsilon| < \pi$
Thus $V_{C_3 \cup C_4}(F) = V_{C_3}(F) + V_{C_4}(F)$ lies between

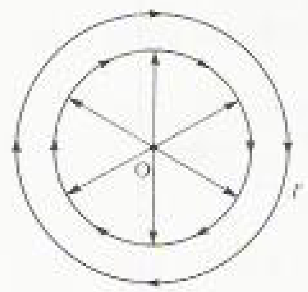
$$3\pi - |\varepsilon| \text{ and } 5\pi + |\varepsilon|$$

(or strictly between 2π and 6π).
Hence $V_{C_3 \cup C_4}(F) = 4\pi$ and F has two zeros in the region $\mathrm{Re}(z) > 0$.
These occur as a complex conjugate pair (since the coefficients of F are real) and hence lie one in the first, and one in the fourth, quadrant. Therefore the remaining two roots lie one in the second and one in the third.

Question 16

(i)





As p describes l once, the 'sharp' end of $V_0(l)$ describes a curve l' of winding number -2 about a fixed point O.

(ii) By the Poincaré theorem, any flow field with a finite number of singularities on a closed orientable surface S has index sum $\chi(S)$. But an orientable surface is determined by χ. Hence S is unique. (In fact, it's a double torus!)

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