Question 13

(i) $w = -\dfrac{(z^2+6)}{2z}$ gives a $1-1$ correspondence of the conic with the z-sphere. The conic is a sphere with no branch point or pinch point.

(ii) $A = 1, B = 2, C = 4$ and so by the main theorem we know that the conic is a pinched sphere.

$$\begin{aligned} w &= \frac{-2z \pm \sqrt{4z^2 - 16(6+z^2)}}{8} \\ &= -\frac{z}{4} \pm \frac{i}{4}\sqrt{3z^2+24} \end{aligned}$$

Thus branch points occur over $z = \pm i\sqrt{8} = \pm 2i\sqrt{2}$, and so at $\left(2i\sqrt{2}, \frac{-i\sqrt{2}}{2}\right)$ and $\left(-2i\sqrt{2}, \frac{+i\sqrt{2}}{2}\right)$. (∞,∞) is a pinch point, by the theorem.

(Alternatively, put $\tilde{z} = \dfrac{1}{z}$, $\tilde{w} = \dfrac{1}{w}$ and obtain $\tilde{w}^2 + 2\tilde{z}\tilde{w} + 4\tilde{z}^2 + 6\tilde{w}^2\tilde{z}^2 = 0$. When $\tilde{z} = 0$, $\tilde{w} = 0$ is a double root and gives the pinch point.)

The conic is a double cover of the z-sphere branched at $(2i\sqrt{2}, -i\sqrt{2}/2)$ and $(-2i\sqrt{2}, +i\sqrt{2}/2)$ with (∞,∞) as a pinch point.

(iii) $A = 1$, $B = 2$, $C = 1$ so $B^2 - 4AC = 0$.
By the theorem the conic is a topological sphere.

$$\begin{aligned} \text{Now } w &= \frac{2z \pm \sqrt{4z^2 - 4(z^2+6z)}}{2} \\ &= -z \pm i\sqrt{6z}. \end{aligned}$$

The only finite branch point is $(z, w) = (0, 0)$.
Since a conic has 0 or 2 branch points, $(z, w) = (\infty, \infty)$ is the other.
The conic is a double cover of the z-sphere branched at (0,0) and (∞,∞).

Question 14

(i) $\chi = 2 - 2g - b$.

(ii) $\tilde{S} \xrightarrow{n} S$ an unbranched cover.

This implies $\beta(S) \leq \beta(\tilde{S}) \leqslant n\beta(S)$, and $\chi(\tilde{S}) = n\chi(S)$.
Now $\beta(S) = 2$ and $\beta(\tilde{S}) = \tilde{b}$.

$$\begin{aligned} \chi(\tilde{S}) = 2 - 2(g+k) - \tilde{b} = n\chi(S) &= n(2 - 2g - 2) \\ &= -2ng. \end{aligned}$$

Hence $\tilde{b} = 2 + 2ng - 2(g+k)$.

So

$$2 \leqslant 2 + 2ng - 2(g+k) \leqslant 2n$$

thus

$$-2ng + 2g \leqslant -2k \leqslant 2n - 2 - 2ng + 2g;$$

so

$$1 + ng - g - n \leqslant k \leqslant ng - g$$

i.e.

$$(g-1)(n-1) \leqslant k \leqslant g(n-1).$$

(iii) If $n = 3$ and $g > 2$ then

$$2(g-1) \leqslant k \leqslant 2g.$$

Hence $\tilde{g} = g + k \geqslant 2(g-1) + g = 3g - 2$.
If $g > 2$ then $3g - 2 > 2g$.