Question 11

(i) $ab = 1; bc = 1; cd = 1 \longrightarrow ab = 1; bd^{-1} = 1$ (eliminating c)
$\longrightarrow ad = 1$ (eliminating b)
$\longrightarrow x = 1$ (writing x for ad)
$\longrightarrow xd^{-1}d = 1$ (introducing the pair $d^{-1}d$)
$\longrightarrow dxd^{-1} = 1$ (cycling)

As the surface is orientable, $\chi = 1$ and $\beta = 1$.

(ii) $abacbdcd = 1$ $\longrightarrow aab^{-1}cbdcd = 1$ (Lemma 1)
$\longrightarrow aab^{-1}d^{-1}b^{-1}ccd = 1$ (Lemma 1)
$\longrightarrow aadb^{-1}b^{-1}ccd = 1$ (Lemma 1)
$\longrightarrow aac^{-1}c^{-1}bbdd = 1$ (Lemma 1)

As the surface is non-orientable, $\chi = -2$ and $\beta = 0$.

(iii) $aba^{-1}b^{-1}cc = 1$ $\longrightarrow abcbac = 1$ (Lemma 1)
$\longrightarrow abbc^{-1}ac = 1$ (Lemma 1)
$\longrightarrow cabbac = 1$ (Lemma 1)
$\longrightarrow abbacc = 1$ (cycling)
$\longrightarrow bbaacc = 1$ (Lemma 1, twice)

As the surface is non-orientable, $\chi = -1$ and $\beta = 0$.

Question 12

(i) The only surface for $\chi = 2$ is the sphere S^2, with $\beta = 0$. It is orientable.

(ii) Now $\chi(pT^2\#bD^2) = 2 - 2p - b$,
$\chi(q\mathbf{RP}^2\#bD^2) = 2 - q - b$.

$\chi = 1$ If $2 - 2p - b = 1$, then $2p + b = 1$ and $p, b \geq 0$.
Hence $p = 0, b = 1$ and we have $S^2\#D^2 = D^2 : \beta = 1$ and it is orientable.
If $2 - q - b = 1$ then $q + b = 1$ and $b \geqslant 0, q > 0$ give $q = 1, b = 0$.

Thus we have $\mathbf{RP}^2 : \beta = 0$ and it is non-orientable.

(iii) $\chi = 0$ If $2 - 2p - b = 0$, then $2p + b = 2$.
So $p = 1, b = 0$ or $p = 0, b = 2$.
This gives $T^2(\beta = 0,$ orientable$)$ or $2D^2(\beta = 2,$ orientable$)$.
If $2 - q - b = 0$, then $q = 2, b = 0$ or $q = 1, b = 1$.

This gives $2\mathbf{RP}^2(\beta = 0$, non-orientable$)$ or $\mathbf{RP}^2\#D^2$ $(\beta = 1$, non-orientable$)$.

(iv) $\chi = -1$ If $2 - 2p - b = -1$, then $2p + b = 3$.
Therefore $p = 0, b = 3$ or $p = 1, b = 1$.
This gives $3D^2(\beta = 3,$ orientable$)$ or $T^2\#D^2(\beta = 1,$ orientable$)$.

If $2 - q - b = -1$, then $q + b = 3$.

Thus $q = 1, b = 2$ or $q = 2, b = 1$ or $q = 3, b = 0$.

This gives $\mathbf{RP}^2\#\,2D^2$ $(\beta = 2$, non-orientable$)$, $2\mathbf{RP}\#D^2$ $(\beta = 1$, non-orientable$)$ and $3\mathbf{RP}^2$ $(\beta = 0$, non-orientable$)$.