Question 4

(i) Let T_1 and T_2 be topological spaces and let $\mathcal{B}$ be a basis for the topology of T_2. Prove that a function $f: T_1 \longrightarrow T_2$ is continuous if $f^{-1}(B)$ is open in T_1 for each set B in $\mathcal{B}$. [4]

(ii) Let $T_1 = \{A, \mathcal{J}_1\}$ where $A = \{a, b, c\}$ and

$$\mathcal{J}_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, A\}.$$

[You need **not** verify that $\mathcal{J}_1$ is a topology on A.]

Prove that if T_2 denotes $\mathbf{R}$ with the usual topology, then a function $g: T_1 \to T_2$ is continuous if and only if $g(b) = g(c)$. [7]

Question 5

For each positive integer m, let

$$B_m = \{1, 3, \ldots, 2m - 1\}.$$

(i) Prove that the collection $\mathcal{B}$ of subsets of the set $\mathbf{N}$ of positive integers which consists of $\mathbf{N}$ together with the sets B_m, $m \in \mathbf{N}$, is a synthetic basis for $\mathbf{N}$. [4]

(ii) Prove that the set E of all even positive integers is closed in the topological space $\{\mathbf{N}, \mathcal{T}\}$, where $\mathcal{T}$ is the topology on $\mathbf{N}$ arising from $\mathcal{B}$ in part (i). [3]

(iii) Determine the closure in $\{\mathbf{N}, \mathcal{T}\}$ of the set $\{2, 6, 7, 8\}$. [4]

Question 6

(i) Prove that each compact subspace of a Hausdorff space T is closed in T. [5]

(ii) For each of the following subspaces of $\mathbf{R}^2$ with the usual topology, state whether or not the subspace is compact. In each case give a brief justification of your answer. [6]

(a) $\{(x_1, 0); x_1 \leqslant 0\}$

(b) $\{(x_1, x_2) : x_1^2 + x_2^2 \leqslant 3\}$

(c) $([0, 1] \times [0, 1]) \cap (Q \times Q)$, where Q is the set of rationals.

Question 7

(i) Assuming that the subspace [0,1] of $\mathbf{R}$ with the usual topology is connected, prove that every path-connected space is connected. [5]

(ii) For each positive integer n, let L_n be the line segment in $\mathbf{R}^2$ joining $\left(\frac{1}{n}, 0\right)$ to $\left(\frac{1}{n}, 1\right)$, let

$$H = \{(x_1, x_2) \in \mathbf{R}^2 : 0 < x_1 \leqslant 1, x_2 = 0\},$$

and let

$$K = \{(0, 1)\} \cup H \cup \bigcup_{n=1}^{\infty} L_n.$$

Prove that the subspace K of $\mathbf{R}^2$ with the usual topology is connected. [6]