Question 1

Solve the following problems:

(i) $u'(x) + u(x) = 0, \quad u(0) = 1;$

(ii) $u''(x) + u(x) = 0, \quad u(0) = 1, \quad u(\pi/2) = 0;$

(iii) $x^2u''(x) + xu'(x) + u(x) = 0 \quad (x > 0), \quad u(1) = 0, \quad u(e) = \sin 1.$ [7]

Question 2

A tornado is modelled as follows. It has a core of radius a rotating at a constant angular frequency ω. Outside the core the motion is that of a vortex of constant strength $2\pi k$, so that the velocity field modelling the tornado, in cylindrical polar coordinates with the vertical axis of the tornado as the z-axis, is

$$\mathbf{u} = \begin{cases} \omega r \mathbf{e}_\phi, & r < a, \\ \dfrac{k}{r}\mathbf{e}_\phi, & r \geq a. \end{cases}$$

(i) Show that this velocity field satisfies the continuity equation for incompressible flow.

(ii) Write down the relation between ω and k that must be satisfied for $\mathbf{u}$ to be everywhere continuous.

(iii) Let $\Gamma = \Gamma(r)$ be the circulation around circles of radius r in planes perpendicular to the z-axis and with centres on that axis.

(a) Show that *outside* the core the motion is irrotational and that the circulation, Γ, is non-zero and constant.

(b) Show also, by finding expressions for the vorticity and for the circulation inside the core, that *inside* the core the vorticity and circulation, Γ, are both non-zero (for $r \neq 0$). [7]

Question 3

(i) Show that, for any incompressible flow, the volume flow rate across a streamline is zero.

(ii) An incompressible, inviscid fluid flows steadily with uniform speed u along a horizontal channel of uniform semi-circular cross-section of radius r.

(a) Express the volume flow rate, V, in terms of u and r, when the fluid fills the channel.

(b) For a given V, show that the relation between the critical speed u_c and the critical depth h_c is given by

$$u_c = \left[gA(h_c) \Big/ \frac{dA}{dh}(h_c)\right]^{\frac{1}{2}}$$

where A is the cross-sectional area of fluid in the channel of depth $h \leq r$. [7]

Question 4

Find the solution $u(r,\theta)$ of Laplace's equation in spherical polar coordinates with cylindrical symmetry, namely

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=0 \qquad (0\le r<R,\ 0\le\theta\le\pi),$$

that satisfies the conditions

$$u(r,\theta) \text{ is bounded and } u(R,\theta)=2\cos^2\frac{\theta}{2}.$$

(The general solution to Laplace's equation in spherical polar coordinates which is periodic in θ may be assumed.) [6]

Question 5

Find, correct to four decimal places, $u(\frac{1}{3}\pi,\frac{1}{2}\pi)$ where $u(x,t)$ satisfies

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2} \qquad (0<x<\infty,\ t>0)$$

subject to the initial conditions

$$u(x,0)=0 \quad (0<x<\infty),$$
$$\frac{\partial u}{\partial t}(x,0)=e^{-x}\cos x \quad (0<x<\infty),$$

and when there is a free-end boundary condition at $x=0$. [6]

Question 6

The 'wave condition' for waves in an incompressible inviscid fluid of infinite depth for which surface tension effects are considered is

$$\omega^2=gk+\frac{Tk^3}{\rho}$$

where T is the magnitude of the surface tension force per unit length, ρ is the density and ω, g and k have their usual meanings.

(i) Using the above wave condition, express the wave speed, c, in terms of the wavelength, λ, and the constants T and ρ.

(ii) Show that $S=Tk^2/(\rho g)$ is a dimensionless parameter.

(iii) Find an expression for the group velocity, c_g, of these waves and hence show that

$$c_g=\frac{g^{1/2}(1+3S)}{2k^{1/2}(1+S)^{1/2}} \quad \text{and} \quad c=\left(\frac{g}{k}\right)^{1/2}(1+S)^{1/2}.$$

(iv) Hence show that when surface tension effects are negligible, so that $S\ll 1$, then the waves are the familiar deep water waves for which $c_g\simeq c/2$. [7]

PART II

Answer **THREE** *questions in this part.*

Each question carries 20% of the total examination marks.

Question 7

(The parts of this question are related.)

(i) Show that $H = p_0/(\rho_0 g)$ has the dimension of length. Here p_0 and ρ_0 are the values of the pressure, p, and the density, ρ, at some datum level $z = z_0$, and g is the magnitude of the acceleration due to gravity. [2]

(ii) Write down the basic equation of fluid statics for a fluid in a gravitational field and the equation of state for a perfect gas. Use these equations to show that both the pressure and the density distributions are exponential functions in z in a static, isothermal, perfect gas in a constant gravitational field. [8]

(iii) Furthermore, show that if both the pressure and the density decrease to a fraction $1/e$ of their respective datum values, p_0 and ρ_0, at $z = z_0$ in a distance H from that datum then $H = p_0/(\rho_0 g)$. (Here e is the base of natural logarithms.) [4]

(iv) Interpret the statement

$H = p_0/(\rho_0 g)$ is known as the height of the constant density atmosphere

by deriving the pressure distribution for a constant-density model and showing how it relates to H. [3]

(v) Comment, briefly, on the modelling of our Earth's atmosphere and the suitability, or otherwise, of constant density, isothermal, or other models. [3]

Question 8

(i) Consider the time-dependent vector field $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ with Cartesian components

$$u_1 = -y \quad \text{and} \quad u_2 = x - 3t.$$

Show that $\mathbf{u}$ can be used to represent the velocity vector field of an incompressible fluid. Explain why it is not possible to define a velocity potential for this flow.
Determine whether or not the flow is steady. [5]

(ii) Write down the equations describing the stream function for the velocity vector field in Part (i). Hence find the stream function for this flow. Sketch some of the streamlines at time $t = 0$, showing the direction of flow. [7]

(iii) Consider the flow of an inviscid fluid of constant density ρ given by the velocity vector field of Part (i) with body force (per unit mass)

$$\mathbf{F} = 3t\,\mathbf{i} - y\,\mathbf{j}.$$

Find the pressure distribution in the fluid (to within an arbitrary function of time) and hence show that the pressure along a streamline, at $t = 0$, is independent of x. [8]

Question 9

(i) Find the general solution $w = w(x, y)$ of the partial differential equation

$$\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x} = 0.$$ [3]

(ii) The function $u(x, y)$ satisfies the partial differential equation

$$3x^2 \frac{\partial^2 u}{\partial y^2} - \frac{1}{12}\frac{\partial^2 u}{\partial x^2} - 6x^2 \frac{\partial u}{\partial y} + \left(x + \frac{1}{12x}\right)\frac{\partial u}{\partial x} = 0 \quad (x \neq 0). \qquad (1)$$

(a) Show that this equation is hyperbolic in the region R of the (x, y)-plane over which it is defined.

(b) Find the equations of the characteristic curves in the region R and hence show that the characteristic coordinates may be chosen to be

$$\zeta = y - 3x^2, \quad \phi = y + 3x^2.$$

(c) Use these characteristic coordinates and the chain rule to transform the partial differential Equation (1) to its standard form.

(d) Hence, and using the result of Part (i), above, find the general solution $u = u(x, y)$ of Equation (1). [17]

Question 10

(i) Show that the Fourier cosine series for the function

$$f(x) = x + \pi$$

on the interval $0 < x < \pi$ is

$$\frac{3\pi}{2} - \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{\cos(2m-1)x}{(2m-1)^2}.$$ [7]

(ii) Show that the eigenvalue problem

$$X''(x) + \lambda X(x) = 0 \qquad (0 < x < \pi)$$
$$X'(0) = X'(\pi) = 0$$

has eigenvalues $\lambda_n = n^2$ $(n = 0, 1, 2, \ldots)$ and corresponding eigenfunctions $X_n(x) = \cos nx$. [4]

(iii) The equation governing the temperature distribution $u(x, t)$ in an insulated bar of length π is given by the diffusion equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad (0 < x < \pi,\ t > 0),$$

where x and t represent distance along the bar and time respectively and k is a positive constant. Initially the temperature distribution of the bar is

$$u(x, 0) = x + \pi \quad (0 < x < \pi).$$

The two ends of the bar are insulated (so that there is no temperature change across the ends).

(a) Write down the boundary conditions at $x = 0$ and $x = \pi$ for the temperature distribution.

(b) Use the method of separation of variables to determine the temperature distribution $u(x, t)$ at times $t > 0$. [9]

Question 11

You have met a variety of fluid models in MST322; for example, for flows along tubes and channels, roller-coating processes, sound waves in a gas, planetary atmospheres and flows around cylindrical and spherical objects. Also you have used a variety of equations, expressed in different coordinate systems, that may be related to the Navier-Stokes equation; for example, the equation of fluid statics, Euler's equation, the wave equation, Bernoulli's equation and the equation of creeping motion.

Use this knowledge to discuss the validity of the following statement:

> The Navier-Stokes equation is fundamental to the study of Newtonian fluid mechanics and has given rise to plausible explanations of various physically observed fluid phenomena.

Your answer should include at least *four* examples of fluid models. For each example:

(i) illustrate how the modelling equation is a special case of, or is related to, the Navier-Stokes equation;

(ii) give both a verbal and a mathematical description;

(iii) indicate at least one physical situation for which the model is appropriate. [20]

[END OF QUESTION PAPER]