Since $z = a|x|^3$, the potential energy is

$$V(x) = mga|x|^3.$$

The kinetic energy is proportional to the square of the speed, v, of the particle, $T = \frac{1}{2}mv^2$. The square of the speed is obtained from Pythagoras' Theorem, as indicated in the figure. We have

$$v^2 = \dot{x}^2 + \dot{z}^2 = \dot{x}^2\left(1 + \left(\frac{dz}{dx}\right)^2\right),$$

and since $dz/dx = 3ax^2$ for $x > 0$ and $dz/dx = -3ax^2$ for $x < 0$, this gives

$$v^2 = \dot{x}^2 + \dot{z}^2 = \dot{x}^2\left(1 + 9a^2x^4\right).$$

Thus the Lagrangian is

$$L(x, \dot{x}) = \tfrac{1}{2}m\dot{x}^2\left(1 + 9a^2x^4\right) - mga|x|^3.$$

(ii) To form the Hamiltonian we first need the generalised momentum p, defined by the relation

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}\left(1 + 9a^2x^4\right), \qquad \text{so} \quad \dot{x} = \frac{p}{m\left(1 + 9a^2x^4\right)}.$$

The general expression relating the Hamiltonian and the Lagrangian is

$$H(x,p) = p\dot{x} - L(x,\dot{x}), \qquad \text{where} \quad p = \frac{\partial L}{\partial \dot{x}},$$

and on using the relation between p and $\dot{x}$ already obtained this gives

$$\begin{aligned} H(x,p) &= \frac{p^2}{m\left(1 + 9a^2x^4\right)} - \left\{\frac{1}{2}\frac{p^2}{m\left(1 + 9a^2x^4\right)} - mga|x|^3\right\} \\ &= \frac{p^2}{2m\left(1 + 9a^2x^4\right)} + mga|x|^3. \end{aligned}$$

(iii) If both x and p are small we may expand this Hamiltonian about the fixed point at the origin and ignore all but the leading terms, to obtain

$$H(x,p) \simeq \frac{1}{2m}p^2\left(1 - 9a^2x^4\right) + mga|x|^3 \simeq \frac{1}{2m}p^2 + mga|x|^3,$$

where we have ignored terms of order p^2x^4.

Note that the period of small oscillations cannot be found by the method of Question 2 in this case, since the approximate Hamiltonian is not of the correct form. However, the Hamiltonian is even in both p and x, so the period, $T(E)$, of the motion with energy E is just four times the time taken to reach the turning point x_1 from $x = 0$; that is,

$$\begin{aligned} T(E) &= 4\int_0^T dt = 4\int_0^{x_1}\frac{dx}{\dot{x}} \\ &= 4m\int_0^{x_1}\frac{dx}{p(x,E)} = 4\sqrt{\frac{m}{2}}\int_0^{x_1}\frac{dx}{\sqrt{E - mgax^3}}, \end{aligned}$$

since

$$p^2 = 2m(E - mgax^3) \quad \text{for } x \geq 0.$$

The turning point, x_1, is the root of $E = mgax^3$, so we can write the integral as

$$T(E) = 4\sqrt{\frac{m}{2E}}\int_0^{x_1}\frac{dx}{\sqrt{1 - (x/x_1)^3}}.$$

On changing variable to $y = x/x_1$ in the integral, the expression for the period can be written in the form

$$T(E) = \frac{4x_1\sqrt{m}}{\sqrt{2E}}J, \qquad \text{where} \quad J = \int_0^1 \frac{dy}{\sqrt{1 - y^3}}.$$

Note that J is just a number, independent of the energy and all other parameters. This expression can be written in the form

$$T(E) = \frac{4(E/m)^{-1/6}J}{2^{1/2}(ag)^{1/3}} = AE^{-1/6}, \qquad \text{where} \quad A = \frac{4Jm^{1/6}}{2^{1/2}(ag)^{1/3}};$$

so $n = 6$.