Question 8

(a) Consider the first-order discrete system

$$x_{n+1} = F(x_n), \qquad \text{where} \quad F(x) = \lambda x + (1-\lambda)x^2,$$

where λ is a parameter.

(i) It is known that $F(x)$ undergoes a sequence of period doubling bifurcations as λ increases, up to a limiting value λ_∞; the first of these occurs at $\lambda = 3$. Briefly describe this process for λ in the range $2 < \lambda < \lambda_\infty$. [5]

(ii) Show that when $\lambda = 4$, the substitution $x_n = \frac{4}{3}\sin^2\theta_n$, where $0 \le \theta_n \le \pi/2$, transforms the system into

$$\sin\theta_{n+1} = \sin(2\theta_n).$$

Describe the principal features of the behaviour of the iteration in this case. [6]

(b) Consider the second-order continuous system with Hamiltonian

$$H(q,p) = \tfrac{1}{2}p^2 + \tfrac{1}{2}q^2 - \tfrac{1}{3}q^3.$$

(i) Show that this system has a stable fixed point at $(0,0)$ and an unstable one at $(0,1)$. Sketch the phase portrait, being careful to indicate the separatrix. [8]

The system is perturbed so that its equations of motion become

$$\dot{q} = p + \epsilon f(q,p), \qquad \dot{p} = -q + q^2,$$

where $f(q,p)$ is some function of the specified arguments, and $0 \le \epsilon \ll 1$.

(ii) Show that the Melnikov function for this system, evaluated along the unperturbed separatrix, is independent of the time, and comment on this result. [6]

[END OF QUESTION PAPER]