of the action I is constant. The averaged motion takes place on the slowly moving ellipse given by the equation

$$p^2 + \omega(t)^2 q^2 = 2\omega(t)I.$$

Thus for short times, that is, times comparable to $2\pi/\omega(t)$, the motion is close to the conservative motion having a 'frozen' value of ω.

But on top of this mean motion there are small oscillations in the action I, which have the same period as the instantaneous 'frozen' motion and amplitude proportional to the rate of change of $\omega(t)$, that is, to ε.

(c) (i) The period of a linear oscillator with Hamiltonian $H = \frac{1}{2}p^2 + \frac{1}{2}\alpha^2 q^2$ is $2\pi/\alpha$, so if ω is slowly halved the period will be slowly doubled.

(ii) From Equation 2 we see that the amplitude of the motion will be increased by a factor of $\sqrt{2}$.

(iii) From Equation 3 we see that the maximum value of the kinetic energy will be halved.

(iv) The total energy is the value of the Hamiltonian, $H = \omega I$, so it will be halved, as the value of the action I is constant.

Question 7

(i) This question concerns the rapid perturbations of a free particle by a non-uniform force. The general theory of this type of system is dealt with in *Unit 10* Section 10.4; the general Hamiltonian considered is defined in Equation 10.27 (page 114), and in this case

$$f(q) = Fqe^{-q^2}.$$

If the applied frequency, Ω, is large, in the sense that during one oscillation of the field the position of the unperturbed particle changes little, we may assume that the perturbed motion comprises two terms. One term, the mean motion, is slowly varying, having the local character of the unperturbed motion, though its global behaviour may be quite different. The other term comprises rapid oscillations of small amplitude. In order to derive the mean motion we make a time-dependent canonical transformation, Equations 10.28 and 10.29 of *Unit 10*, to a coordinate system which models this behaviour locally, as it is derived from a good local approximation, and average the resulting (exact) Hamiltonian over one period of the perturbing force.

If (Q, P) are the mean motion coordinates, the mean motion Hamiltonian is given in Exercise 10.13, Equation 10.34; in this case we obtain

$$\overline{K}(Q,P) = \frac{P^2}{2m} + \frac{F^2}{4m\Omega^2}\left(1-2Q^2\right)^2 e^{-2Q^2} \qquad (4)$$

since

$$f'(Q) = (1-2Q^2)e^{-Q^2}.$$

(ii) The effective potential is proportional to the function

$$g(Q) = \left(1-2Q^2\right)^2 e^{-2Q^2}.$$

This function tends to zero as $Q \to \pm\infty$; it is a positive, even, continuous function so has a local maximum at $Q = 0$. The stationary points are at the roots of

$$g'(Q) = 4Q(2Q^2-3)(1-2Q^2)e^{-2Q^2} = 0$$

which are $Q^2 = 1/2$, which are also the zeros of $g(Q)$, $Q = 0$ and $Q^2 = 3/2$, which must be local maxima. Since

$$g(0) = 1 > g\left(\sqrt{3/2}\right) = 4e^{-3},$$

the maximum in the effective potential at the origin is higher than the maxima at $Q^2 = 3/2$. The sketch of the effective potential in Figure 5 shows these features.