Question 4

(i) A canonical transformation is defined in terms of the $F_2(P,q)$ generating function by the relations

$$Q = \frac{\partial F_2}{\partial P} \quad \text{and} \quad p = \frac{\partial F_2}{\partial q}.$$

So by comparison with the given relation we have

$$\frac{\partial F_2}{\partial q} = p = \frac{P}{\sqrt{q}}.$$

Integration gives

$$F_2(P,q) = 2P\sqrt{q}.$$

Note we have set the arbitrary function of P, arising from the integration, to zero as we are asked for *a* conjugate variable, not *all* possible conjugate variables.

From this generating function we have

$$Q = \frac{\partial F_2}{\partial P} = 2\sqrt{q}, \quad \text{or} \quad q = \tfrac{1}{4}Q^2.$$

(ii) The Hamiltonian in the new representation is just the old Hamiltonian expressed in terms of the new coordinates (though this would not be true if the canonical transformation were time-dependent; see *Unit 6* Section 6.5). Thus since

$$p = \frac{P}{\sqrt{q}} = \frac{2P}{Q},$$

we have

$$H(Q,P) = \tfrac{1}{2}\left(\frac{2P}{Q}\right)^2 - \frac{4}{Q^2} = \frac{2P^2}{Q^2} - \frac{4}{Q^2} = \frac{2}{Q^2}(P^2-2).$$

(iii) The system is conservative so the phase curves are the contours of the Hamiltonian; these are

$$2P^2 - EQ^2 = 4. \tag{1}$$

For negative energies, $-E = W > 0$ say, this equation becomes $2P^2 + WQ^2 = 4$ which is the equation of an ellipse crossing the P-axis at $P^2 = 2$, for all W, and crossing the Q-axis at $Q^2 = 4/W$. Some of these ellipses are shown in Figure 4.

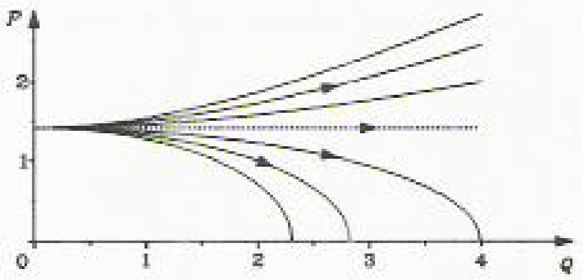

Figure 4 Some representative contours for the Hamiltonian $H(Q,P) = 2(P^2-2)/Q^2$; only positive momenta are shown, as the contours are symmetric with respect to reflections in the Q-axis.

For positive energies, $E > 0$, Equation 1 is the equation of a hyperbola which intersects the P-axis at $P = \sqrt{2}$, regardless of the value of E.

The phase curve for $E = 0$ is the separatrix, being the boundary between these two types of motion. Thus the equation of the separatrix is $P = \pm\sqrt{2}$, and one branch of it is shown by the dotted line in the figure.

(It is interesting to note that a phase point which approaches the P-axis with $P < 0$ flips to positive P instantaneously at $Q = 0$; this strange behaviour is a consequence of the relation between Q and q, namely $Q = 2\sqrt{q}$, being non-differentiable at $q = 0$: since $\dot{Q} = \dot{q}q^{-1/2}$ we see that $\dot{Q}$ does not exist at $Q = 0$. This is true whatever the energy of the phase curve. It follows that the negative energy motion is bound and periodic, while positive energy motion is unbounded.)