(c) In the angle-action variables of the unperturbed system, this Hamiltonian is

$$H(\theta, I) = I + \frac{\varepsilon}{1+\sqrt{2I}\cos\theta}.$$

The first-order correction to the Hamiltonian in the angle-action variables (ϕ, J) of the perturbed system is just the mean value of the perturbation taken over an unperturbed orbit:

$$\begin{aligned}\overline{H}_1(J) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\phi}{1+\sqrt{2I}\cos\phi}\\ &= \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{1+\sqrt{2I}\cos\phi}\\ &= \frac{1}{\sqrt{1-2J}}, \qquad \text{for} \quad 2J<1,\end{aligned}$$

where we have used a definite integral given in the Handbook. (If $2J \geq 1$ the integral does not exist and neither does the perturbation expansion.)

The new and old angle-action variables are related by a generator $G(\phi, J)$, defined in *Unit 8* Section 8.4, Equation 8.56. For this problem we have

$$\begin{aligned}\frac{\partial G}{\partial \phi} &= -\frac{1}{1+\sqrt{2I}\cos\phi} + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{d\phi}{1+\sqrt{2I}\cos\phi}\\ &= \frac{1}{\sqrt{1-2J}} - \frac{1}{1+\sqrt{2I}\cos\phi}.\end{aligned}$$

This last equation for G can be integrated using an integral in the Handbook; we obtain

$$\begin{aligned}G(\phi, J) &= \frac{\phi}{\sqrt{1-2J}} - \int_0^{\phi}\frac{d\phi}{1+\sqrt{2I}\cos\phi}\\ &= \frac{\phi}{\sqrt{1-2J}} - \frac{2}{\sqrt{1-2J}}\tan^{-1}\left\{\sqrt{\frac{1-\sqrt{2J}}{1+\sqrt{2J}}}\tan\left(\frac{\phi}{2}\right)\right\}\\ &= \frac{2}{\sqrt{1-2J}}\left\{\frac{\phi}{2} - \tan^{-1}\left(\sqrt{\frac{1-\sqrt{2J}}{1+\sqrt{2J}}}\tan\left(\frac{\phi}{2}\right)\right)\right\}.\end{aligned}$$

Question 6

(a) The Hamiltonian of the system is

$$H(q,p) = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 q^2.$$

Since the phase curves are contours of the Hamiltonian, and since the equation $H(q,p) = E$ is that of an ellipse, it follows that the phase curves are elliptical. The curve of energy E crosses the q-axis at $q^2 = 2E/\omega^2$ and the p-axis at $p^2 = 2E$, so the area of the ellipse is $A(E) = 2\pi E/\omega$. Since the action is defined by the relation $I(E) = A(E)/2\pi$ (*Unit* 7 Subsection 7.3.2), this gives $I(E) = E/\omega$. Thus it follows that the ellipse with action I crosses the q- and p-axes at $\sqrt{2I/\omega}$ and $\sqrt{2I\omega}$ respectively.

The maximum amplitude of the motion is just the value of q at which the ellipse crosses the q-axis,

$$\max(q) = \sqrt{2I/\omega}. \tag{2}$$

The maximum kinetic energy, T, is at the point where the potential energy, $V(q) = \frac{1}{2}\omega^2 q^2$, is zero, that is, where the ellipse crosses the p-axis:

$$\max(p) = \sqrt{2I\omega} \qquad \text{so} \qquad \max(T) = E = \omega I. \tag{3}$$

(b) If ω changes very little over the time $2\pi/\omega$, that is, over one unperturbed period of the motion, and changes by a quantity of order unity in a time ε^{-1}, where ε is a small quantity, then over the time interval $(0, \varepsilon^{-1})$ the mean value