PART I

You should answer at least one question from this part.

Question 1

(a) Explain how the linearisation of a non-linear autonomous second-order system

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x})$$

about a fixed point $\mathbf{x}_f$ can be used to classify the fixed point. In your answer make sure to state any circumstances under which linearisation fails to give a reliable guide to the nature of the motion of the original system in the neighbourhood of $\mathbf{x}_f$. [*10*]

(b) Find all the fixed points of the following non-linear system:

$$\dot{x} = 3xy, \qquad \dot{y} = 4 - x^2 - 4y^2.$$

In each case, say whether the fixed point is stable or unstable, if it is possible to do so on the basis of linearisation. [*15*]

Question 2

(i) Sketch the graph of the function

$$V(q) = qe^{-q^2}.$$ [*6*]

(ii) Locate and classify the fixed points of the Hamiltonian

$$H(q,p) = \tfrac{1}{2}p^2 + qe^{-q^2},$$

and sketch some representative contours of it, including any separatrixes. [*12*]

(iii) Find the period of small oscillations about any stable fixed points found in part (ii). [*7*]

Question 3

A particle of mass m slides under gravity on a smooth rigid wire in the vertical (x, z)-plane, where the z-axis is vertically upwards. The shape of the wire is defined by

$$z = a|x|^3,$$

where a is a positive constant.

(i) Show that the Lagrangian of the system is

$$L(x, \dot{x}) = \tfrac{1}{2}m\dot{x}^2\left(1 + 9a^2x^4\right) - mga|x|^3.$$ [*8*]

(ii) Show that the Hamiltonian of the system is

$$H(x,p) = \frac{p^2}{2m\left(1 + 9a^2x^4\right)} + mga|x|^3.$$ [*5*]

(iii) Show that the period of small oscillations of the system about $x = 0$ is given approximately by

$$T(E) \simeq 4\sqrt{\frac{m}{2E}}\int_0^{x_1} \frac{dx}{\sqrt{1 - (x/x_1)^3}},$$

where E is the energy of the motion and $x_1 > 0$ the turning point. Deduce that the period $T(E)$ of small oscillations with energy E takes the form

$$T(E) \simeq AE^{-1/n},$$

where A is a positive constant and n is an integer whose value is to be found. [*12*]