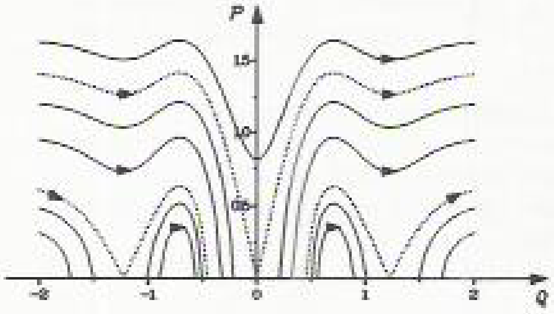

Figure 6 Some representative contours of the Hamiltonian given in Equation 4; the contours for negative P are obtained by reflection in the Q-axis.

(iii) We have already shown that the particle will be turned back unless v_0 satisfies inequality 5. It follows that the required condition is

$$v_0 > \frac{F}{m\Omega\sqrt{2}}.$$

Question 8

(a) (i) This is a quadratic one-dimensional map so will behave in the same manner as that treated in *Unit 12* Section 12.3, $y_{n+1} = y_n^2 + c$. On putting $x = ay + b$, the given map becomes

$$ay_{n+1} = b(b-1)(1-\lambda) + ay_n\{\lambda + 2b(1-\lambda)\} + a^2(1-\lambda)y_n^2,$$

so on choosing

$$b = \frac{\lambda}{2(1-\lambda)}, \quad a = \frac{1}{1-\lambda} \quad \text{and} \quad c = b(b-1)(1-\lambda) = \tfrac{1}{4}\lambda(2-\lambda),$$

we regain the map $y_{n+1} = y_n^2 + c(\lambda)$.

This latter map undergoes a sequence of period doubling bifurcations, starting at $c = -\frac{3}{4}$, corresponding to $\lambda = 3$, and ending at $c_\infty \simeq -1.39$, corresponding to $\lambda_\infty = 1 + \sqrt{1 - 4c_\infty}$, at the parameter values $c_k = c(\lambda_k)$, $k = 1, 2, \ldots$, and for large k these satisfy the relation

$$\lambda_k \simeq \lambda_\infty + \alpha\delta^{-k}, \qquad \text{large } k,$$

where δ is a universal number and α is a constant, but is not universal.

At each period doubling bifurcation, a periodic orbit which was previously stable becomes unstable, and a new periodic orbit of twice the period is born. Thus the system has attracting periodic orbits, successively of periods 1, 2, 4, 8, 16 and so on.

(ii) On putting $\lambda = 4$ and $x = \frac{4}{3}\sin^2\theta$, we obtain

$$\begin{aligned} F(x) &= \tfrac{16}{3}\sin^2\theta - \tfrac{16}{3}\sin^4\theta \\ &= \tfrac{16}{3}\sin^2\theta(1-\sin^2\theta) = \tfrac{16}{3}\sin^2\theta\cos^2\theta = \tfrac{4}{3}\sin^2 2\theta, \end{aligned}$$

and hence

$$\sin^2\theta_{n+1} = \sin^2 2\theta_n$$

or $\quad \sin\theta_{n+1} = \sin 2\theta_n, \qquad \text{since} \quad 0 \le \theta_n \le \pi/2.$

This defines the tent map, and the three distinctive features of its motion are:

- there are periodic orbits of every period $1, 2, 3, \ldots$;
- there are infinitely many dense orbits;
- it shows extreme sensitivity to initial conditions.