Question 5

(a) The easiest way to proceed with this type of equation is first to put $\epsilon = 0$ in order to obtain the zero-order solution. In this case we obtain, when $\epsilon = 0$, $x = \theta$. Thus the perturbation expansion will be of the form

$$x = \theta + \epsilon x_1 + \epsilon^2 x_2 + O(\epsilon^3).$$

On substituting this into the equation we obtain

$$\theta = \theta + \epsilon x_1 + \epsilon^2 x_2 + \epsilon \sinh(\theta + \epsilon x_1) + O(\epsilon^3).$$

But on using the first-order Taylor expansion we have

$$\sinh(\theta + \epsilon x_1) = \sinh\theta + \epsilon x_1 \cosh\theta + O(\epsilon^2),$$

so the equation becomes

$$\begin{aligned}\theta &= \theta + \epsilon x_1 + \epsilon^2 x_2 + \epsilon\left(\sinh\theta + \epsilon x_1 \cosh\theta\right) + O(\epsilon^3)\\ &= \theta + \epsilon\left(x_1 + \sinh\theta\right) + \epsilon^2\left(x_2 + x_1\cosh\theta\right) + O(\epsilon^3).\end{aligned}$$

On equating the coefficients of the powers of ϵ this last equation gives

$$x_1 = -\sinh\theta \quad \text{and} \quad x_2 = -x_1\cosh\theta = \sinh\theta\cosh\theta = \tfrac{1}{2}\sinh 2\theta.$$

Thus the perturbation expansion for x is

$$x = \theta - \epsilon\sinh\theta + \tfrac{1}{2}\epsilon^2\sinh 2\theta + O(\epsilon^3).$$

(b) As in the first part of the question we proceed by setting $\epsilon = 0$ to find the zero-order solution, that is, the solution of the equation

$$\frac{dx}{dt} = \frac{1}{x} \quad \text{with} \quad x(0) = 1.$$

This equation is separable, so can be written in the form

$$\int_1^x dx\, x = \int_0^t dt, \qquad \text{or} \quad x^2 = 2t + 1.$$

Now make the perturbation expansion

$$x(t) = x_0(t) + \epsilon x_1(t) + O(\epsilon^2),$$

where $x_0(t) = \sqrt{2t+1}$, as found above. On substituting this expansion into the differential equation we obtain

$$\begin{aligned}\dot{x}_0 + \epsilon\dot{x}_1 &= \frac{1}{x_0 + \epsilon x_1} + \frac{\epsilon}{x_0^{3/2}} + O(\epsilon^2)\\ &= \frac{1}{x_0}\left(1 - \epsilon\frac{x_1}{x_0}\right) + \frac{\epsilon}{x_0^{3/2}} + O(\epsilon^2)\\ &= \frac{1}{x_0} - \epsilon\left(\frac{x_1}{x_0^2} - \frac{1}{x_0^{3/2}}\right).\end{aligned}$$

On equating the coefficients of the powers of ϵ, the following two differential equations are obtained:

$$\dot{x}_0 = \frac{1}{x_0}, \quad x_0(0) = 1,$$

$$\dot{x}_1 = -\frac{x_1}{x_0^2} + \frac{1}{x_0^{3/2}}, \quad x_1(0) = 0.$$

The first of these is just the zero-order equation treated above, so $x_0(t) = \sqrt{2t+1}$, and so we can write the second equation in the form

$$\frac{dx_1}{dt} + \frac{x_1}{2t+1} = \frac{1}{(2t+1)^{3/4}}.$$

This equation can be solved by using an integrating factor:

$$\frac{dx_1}{dt} + \frac{x_1}{2t+1} = \frac{1}{\sqrt{2t+1}}\frac{d}{dt}\left(x_1\sqrt{2t+1}\right).$$

Thus the equation can be integrated as

$$x_1\sqrt{2t+1} = \int_0^t dt\,(2t+1)^{-1/4} = \left[\tfrac{2}{3}(2t+1)^{3/4}\right]_0^t = \tfrac{2}{3}\left((2t+1)^{3/4} - 1\right),$$

so we have

$$x(t) = (2t+1)^{1/2} + \tfrac{2}{3}\epsilon\left((2t+1)^{1/4} - (2t+1)^{-1/2}\right) + O(\epsilon^2).$$