- At the energy $E = (2e)^{-1/2}$ there is another separatrix, emanating from the unstable fixed point at $(1/\sqrt{2}, 0)$. This separatrix has four branches, only two of which are shown in the figure (the others being the reflections of these in the q-axis). This separatrix divides motion which runs from $q = -\infty$ to $q = \infty$, and vice versa, from motion which is turned back by the potential barrier.
- For energies $E > (2e)^{-1/2}$, that is, $E > \max(V(q))$, the motion can pass over the potential barrier; the distinctive feature of this motion is that $\dot{q} \neq 0$, so as $t \to \infty$ either $q \to \infty$ or $q \to -\infty$, depending upon the initial sign of $p = \dot{q}$.

(iii) We find the motion in the region of the stable fixed point at $q = q_m = -1/\sqrt{2}$ by expanding the potential about this point. Put $q = q_m + x$ and ignore terms of $O(x^3)$. Then

$$V(q_m + x) \simeq V(q_m) + (q_m - x)\frac{dV}{dq} + \tfrac{1}{2}(q_m - x)^2 \frac{d^2V}{dq^2},$$

where all derivatives are evaluated at q_m. But the first term is constant so may be ignored; and $dV/dq = 0$ at q_m, by definition. So the local approximation to the Hamiltonian is

$$H(x, p) = \tfrac{1}{2}p^2 + \tfrac{1}{2}(q_m - x)^2 \frac{d^2V}{dq^2}.$$

The result quoted in the Handbook for the Hamiltonian

$$H(q, p) = \tfrac{1}{2}A^2p^2 + \tfrac{1}{2}B^2q^2$$

shows that the frequency of oscillatory motion is

$$\omega = \sqrt{\frac{d^2V}{dq^2}}.$$

So all we need now is the second derivative of the potential at its minimum. This is

$$\begin{aligned}\frac{d^2V}{dq^2} &= 2q(2q^2 - 3)e^{-q^2} \\ &= 2\sqrt{2/e} \qquad \text{at} \quad q = 1/\sqrt{2}.\end{aligned}$$

Thus

$$\omega = 2^{3/4}e^{-1/4}.$$

Question 3

(i) The Lagrangian is given by the general expression

$$L(x, \dot{x}) = T(x, \dot{x}) - V(x),$$

where $T(x, \dot{x})$ is the kinetic energy and $V(x)$ the potential energy, both expressed in terms of the generalised coordinate x and its generalised velocity $\dot{x}$.

The potential energy is just mgz, as z is the height of the particle above a fixed point, which we chose to be $z = 0$, as shown in Figure 3; note that the position of this reference point is immaterial, as changing it only changes the potential energy by a constant.

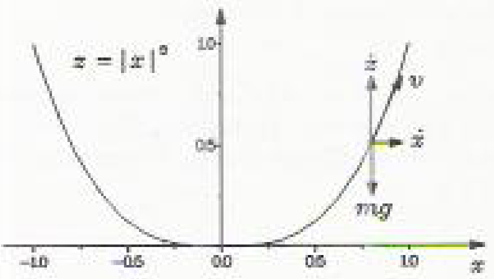

Figure 3 Graph of the potential energy $V(x) = a|x|^3$, with $a = 1$.