Question 6

(a) A particle of unit mass moves under the influence of a linear attractive force of magnitude $\omega^2 q$, where ω is a constant. Show that its phase curve of action I is an ellipse the length of whose axes are $\sqrt{2I\omega}$ and $\sqrt{2I/\omega}$. What are the amplitude of the particle's motion, and the maximum value of its kinetic energy, in terms of its action? [7]

Consider now what happens when ω is no longer constant but changes slowly with time; the change may be assumed to be adiabatic.

(b) Describe briefly and qualitatively the nature of the motion of the phase point representing the particle, with particular reference to the elliptical phase curves of the unperturbed Hamiltonian. How does the action change, on average, and in detail? [10]

(c) Suppose that over a sufficiently long time the value of ω is slowly halved. How do the following quantities change in this time?

(i) The period of the motion. [2]

(ii) The amplitude of the motion. [2]

(iii) The particle's maximum kinetic energy. [2]

(iv) The particle's total energy. [2]

Question 7

A particle is moving subject to a rapidly varying force, with the Hamiltonian

$$H(q,p,t) = \frac{p^2}{2m} + Fqe^{-q^2}\sin\Omega t,$$

where m is the particle's mass, F is a constant, and the frequency Ω of the applied force is large.

(i) Show that the mean motion Hamiltonian is

$$\overline{K}(Q,P) = \frac{P^2}{2m} + \frac{F^2}{4m\Omega^2}\left(1-2Q^2\right)^2 e^{-2Q^2},$$

where (Q,P) are the mean motion coordinates. Explain briefly how this Hamiltonian is derived. [9]

(ii) Give a brief qualitative description of the motion in phase space under the Hamiltonian of part (i), including in your answer a sketch of the effective potential as a function of Q. [12]

(iii) Show that a particle moving towards the origin, but starting a long way away from it, will be turned back unless its initial speed v_0 satisfies

$$v_0 > \frac{F}{m\Omega\sqrt{2}}.$$ [4]