MS323 Solutions to Specimen Examination Paper

Question 1

(a) If a fixed point $\mathbf{x}_f$ is simple then it may be classified by first making a Taylor expansion of the velocity function about $\mathbf{x}_f$, ignoring all second-order terms:

$$v_x(\mathbf{x}) = v_x(\mathbf{x}_f) + (x - x_f)\frac{\partial v_x}{\partial x}(\mathbf{x}_f) + (y - y_f)\frac{\partial v_x}{\partial y}(\mathbf{x}_f) + O\left((\mathbf{x} - \mathbf{x}_f)^2\right),$$

$$v_y(\mathbf{x}) = v_y(\mathbf{x}_f) + (x - x_f)\frac{\partial v_y}{\partial x}(\mathbf{x}_f) + (y - y_f)\frac{\partial v_y}{\partial y}(\mathbf{x}_f) + O\left((\mathbf{x} - \mathbf{x}_f)^2\right).$$

But since $\mathbf{x}_f$ is a fixed point, $\mathbf{v}(\mathbf{x}_f) = \mathbf{0}$. Now move the origin to the fixed point by the change of variables $\mathbf{z} = \mathbf{x} - \mathbf{x}_f$. This leads to the following linear equations of motion:

$$\dot{\mathbf{z}} = \begin{bmatrix} \dfrac{\partial v_x}{\partial x} & \dfrac{\partial v_x}{\partial y} \\ \dfrac{\partial v_y}{\partial x} & \dfrac{\partial v_y}{\partial y} \end{bmatrix} \mathbf{z}, \qquad \mathbf{z} = \mathbf{x} - \mathbf{x}_f,$$

which can be written in the matrix form

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z},$$

where, for a simple fixed point, $\mathbf{A}$ is a constant non-singular matrix.

The linearisation theorem states that the flows of the non-linear system and the linear system are equivalent in a neighbourhood of a simple fixed point, provided that the linear system does not have a centre, in which case the linear system does not indicate how the non-linear system behaves.

The linear system can be simplified by making a linear transformation to new coordinates $\mathbf{u} = \mathbf{B}\mathbf{z}$ to cast the equations into the form

$$\dot{\mathbf{u}} = \mathbf{D}\mathbf{u},$$

where $\mathbf{D} = \mathbf{B}\mathbf{A}\mathbf{B}^{-1}$ is another matrix which has one of only three possible forms. Which one of these forms is taken by $\mathbf{D}$ depends only upon the eigenvalues of the original matrix $\mathbf{A}$; these, in turn, depend only upon the trace and the determinant of $\mathbf{A}$.

Thus the form of the phase curves of the non-linear system is determined by $\operatorname{tr}\mathbf{A}$ and $\det\mathbf{A}$, provided that the fixed point is simple (so $\det\mathbf{A} \neq 0$), and provided that the linear system is not a centre, that is, that the eigenvalues of $\mathbf{A}$ are not purely imaginary.

(b) The fixed points are at the roots of

$$xy = 0 \quad \text{and} \quad x^2 + 4y^2 = 4,$$

that is,

$$x = 0 \quad \text{and} \quad y = \pm 1$$

and

$$y = 0 \quad \text{and} \quad x = \pm 2.$$

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