PART II

You should answer at least one question from this part.

Question 5

(a) Find, correct to order ε^2, the solution to the equation

$$x + \varepsilon \sinh x = \theta,$$

where ε is a small positive parameter and θ a fixed parameter of $O(1)$. [5]

(b) Find an approximate solution, accurate to first order in ε, to the differential equation

$$\dot{x} = \frac{1}{x} + \frac{\varepsilon}{x^{3/2}},$$

with initial condition $x(0) = 1$. [10]

(c) Consider the Hamiltonian

$$H(q,p) = H_0(q,p) + \varepsilon H_1(q) = \tfrac{1}{2}p^2 + \tfrac{1}{2}q^2 + \frac{\varepsilon}{1+q},$$

where ε is a small constant. The angle-action variables of the unperturbed system are (θ, I), those of the perturbed system are (ϕ, J). Show that for $J < \frac{1}{2}$,

$$\int_{-\pi}^{\pi} d\theta \, H_1(\theta, I) = \frac{2\pi}{\sqrt{1-2I}},$$

and deduce that, to first order in ε,

$$I = J + \varepsilon\frac{\partial G}{\partial \phi}, \quad \theta = \phi - \varepsilon\frac{\partial G}{\partial J},$$

where

$$G(\phi, J) = \frac{2}{\sqrt{1-2J}}\left\{\frac{\phi}{2} - \tan^{-1}\left(\sqrt{\frac{1-\sqrt{2J}}{1+\sqrt{2J}}}\tan(\phi/2)\right)\right\}.$$ [10]

[Hint: You may assume that the angle-action variables, (θ, I), of the unperturbed system, $\varepsilon = 0$, are given by

$$q = \sqrt{2I}\cos\theta, \qquad p = \sqrt{2I}\sin\theta.]$$