(ii) The fixed points of the Hamiltonian

$$H(q,p) = \tfrac{1}{2}p^2 + qe^{-q^2}$$

are at the points where the Hamiltonian function is stationary, that is, at the roots of

$$\frac{\partial H}{\partial p} = p = 0, \qquad \frac{\partial H}{\partial q} = \frac{dV}{dq} = 0.$$

In other words, the fixed points of the system are at $p = 0$ and at the stationary points of the potential, found in part (i).

Since the contours of $H(q,p)$ are the phase curves, those fixed points which are extrema (maxima and minima) of $H(q,p)$ must be stable, as in their vicinity the contours are closed curves. On the other hand, at the saddle points of $H(q,p)$ the phase curves do not remain close to the fixed point, so these are unstable fixed points.

Now in the p direction, the given Hamiltonian has a global minimum at $p = 0$, so its extrema can only be minima; this is always true of Hamiltonians of mechanical type. The minima of the Hamiltonian function occur where the potential has a minimum. At the maxima of the potential, the Hamiltonian has a saddle.

Thus the system has a stable centre at $(-1/\sqrt{2}, 0)$ and an unstable saddle at $(1/\sqrt{2}, 0)$.

Some representative phase curves are shown in Figure 2.

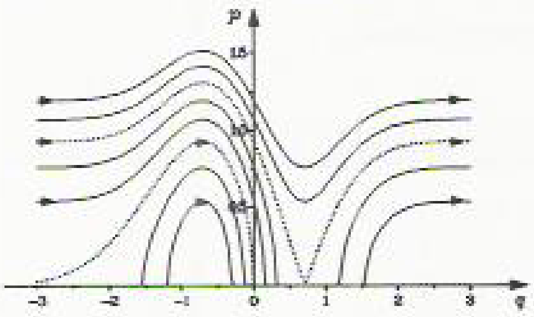

Figure 2 Some representative contours of the Hamiltonian $H = \frac{1}{2}p^2 + qe^{-q^2}$.

The important features shown in this figure are discussed below.

- About the stable fixed point at $(-1/\sqrt{2}, 0)$ the contours of $H(q,p)$ are closed; these represent librational motion, that is, motion which has bounded variation and which is periodic.

  The minimum energy for this type of motion is

  $$\min(E) = -(2e)^{-1/2}$$

  and the maximum energy is $\max(E) = 0$.

- At the energy $E = 0$ there is a separatrix, shown by the dotted line through the origin in the figure. This phase curve represents motion whose momentum tends to zero as $q \to -\infty$; on one side of this curve there is bound periodic motion and on the other side unbounded motion.

- There are two types of motion with energy in the range

  $$0 < E < \max(V(q)) = (2e)^{-1/2}.$$

  There is motion confined to the region $q < 1/\sqrt{2}$ and motion confined to the region $q > 1/\sqrt{2}$. In both cases the motion is unbounded, but in neither case does the motion have sufficient energy to surmount the potential maximum at $q = 1/\sqrt{2}$.