Question 3

(a) A dynamical system may be represented either in (q, p)-space, in terms of a Hamiltonian function $H(q, p, t)$, or in $(q, \dot{q})$-space, in terms of a Lagrangian function $L(q, \dot{q}, t)$. Explain the connection between the two representations; include in your explanation a geometrical description of the relationship between the coordinates. How is the Hamiltonian found when the Lagrangian is known? [5]

(b) Show that

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}.$$ [4]

(c) Find the Hamiltonian corresponding to the Lagrangian

$$L(q, \dot{q}, t) = \tfrac{1}{2}\dot{q}^2 - f(q + bt)$$ [5]

where b is a positive constant, $f(x)$ an arbitrary function and where $f(q + bt)$ is the value of this function evaluated at $x = q + bt$.

(d) By defining a new generalised coordinate $Q = q + bt$ find a time-dependent canonical transformation which converts the Hamiltonian found in part (c) into the conservative Hamiltonian

$$K(Q, P) = \tfrac{1}{2}(P + b)^2 + f(Q).$$ [7]

(e) For the case $f(x) = \frac{1}{2}x^2$ solve Hamilton's equations formed from K and hence find $q(t)$. [4]

Question 4

(a) Show that the F_1 generating function for the time-dependent canonical transformation

$$q = Q\cos t + P\sin t, \quad p = -Q\sin t + P\cos t$$

is

$$F_1(Q, q, t) = \frac{(Q^2 + q^2)\cos t - 2Qq}{2\sin t}$$ [6]

(b) Verify

$$\frac{\partial F_1}{\partial t} = -\tfrac{1}{2}\left(P^2 + Q^2\right),$$

and hence show that the Hamiltonian

$$H(q, p) = \tfrac{1}{2}\left(p^2 + q^2\right) + \varepsilon q^4$$

is transformed to

$$K(Q, P, t) = \varepsilon\left(Q\cos t + P\sin t\right)^4.$$ [8]

(c) If $|\varepsilon|$ is small provide an argument to justify replacing K by its average over the time interval $(0, 2\pi)$, and show that this average is

$$\overline{K} = \tfrac{3}{8}\varepsilon\left(Q^2 + P^2\right)^2.$$ [5]

(d) By solving for this mean motion obtain an approximation expression for $q(t)$. Using your solution describe how this motion differs from that with $\varepsilon = 0$. [6]

Hint: in part (d) you may find it helpful to use the angle-action variables

$$Q = \sqrt{2I}\sin\theta, \quad P = \sqrt{2I}\cos\theta.$$