Question 6

(a) Consider a particle of mass m moving in the potential

$$V(q) = a|q|^n,$$

where a is a positive constant and n a positive integer. Show that the action, I, and the energy, E, are related by

$$I = \frac{2\sqrt{2m}}{\pi}\left(\frac{E^{n+2}}{a^2}\right)^{\frac{1}{2n}} J_n, \quad J_n = \int_0^1 dy\,\sqrt{1-y^n}. \qquad [6]$$

(b) Now consider a particle of unit mass moving in the potential

$$V(q,t) = A(t)q^2 + B(t)|q|,$$

where $A(t)$ and $B(t)$ are two slowly varying functions, such that $A(t)$ slowly decreases from 1 to 0 and $B(t)$ slowly increases from 0 to 1 over some long time T; thus

$$A(t) = \begin{cases} 1 & t \le 0 \\ 0 & t \ge T \end{cases} \qquad B(t) = \begin{cases} 0 & t \le 0 \\ 1 & t \ge T \end{cases}.$$

You may assume that the change in the system is adiabatic.

Describe, briefly and qualitatively, the nature of the motion of the phase point representing the particle; you should include a description of how the phase curve on which the phase point moves for $t \le 0$ evolves into the one on which it moves for $t \ge T$. [7]

(c) By calculating the action for $t \le 0$ and for $t \ge T$, show that if the particle's initial energy is E_i and its final energy is E_f then

$$E_f = \left(\frac{3\pi}{8}E_i\right)^{2/3}. \qquad [8]$$

(d) Find the relation between the amplitudes of the motion for $t < 0$ and for $t > T$. [4]

Question 7

A particle of mass m is moving subject to a time-dependent force, the Hamiltonian being

$$H(q,p,t) = \frac{p^2}{2m} - \tfrac{1}{2}m\omega^2 q^2\left(1 + F\sin\Omega t\right) + \tfrac{1}{4}a^4 q^4$$

where F, ω and a are constants, and the frequency Ω is large.

(a) Show that the mean motion Hamiltonian is given in terms of the mean motion coordinates (Q, P) by

$$\overline{K}(Q,P) = \frac{P^2}{2m} - \frac{1}{2}m\omega^2 Q^2\left(1 - \frac{\omega^2 F^2}{2\Omega^2}\right) + \frac{1}{4}a^4 Q^4.$$

Give a brief account of the approximations used in deriving this Hamiltonian. [9]

(b) Provide a sketch of the effective potential as a function of Q and hence give a brief qualitative description of the motion in phase space, being careful to distinguish between the cases

$$F < \frac{\Omega}{\omega}\sqrt{2} \quad \text{and} \quad F > \frac{\Omega}{\omega}\sqrt{2}. \qquad [10]$$

(c) Initially $F > \Omega\sqrt{2}/\omega$ and F is allowed to decrease very slowly to zero. Provide a qualitative description of the possible types of final motion after F has reached zero. [6]