PART II

You should answer at least one question from this part.

Question 5

(a) Find, correct to order ε^2, the solution nearest $x = a$ to the equation

$$x^3 + \varepsilon x^2 - a^3 = 0,$$

where a is a positive constant and ε is a small parameter. [5]

(b) Consider the equation

$$\frac{dx}{dt} = -\lambda x + \varepsilon x(\alpha t)$$

with initial condition $x(0) = A$, where A, α and λ are positive constants and where ε is a small parameter. Note that $x(\alpha t)$ means the function $x(t)$ evaluated at αt.

Show that if $x(t)$ is a solution of this equation successive terms of the perturbation expansion

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \cdots$$

are related by the equation

$$x_n(t) = e^{-\lambda t} \int_0^t dz\, e^{\lambda z} x_{n-1}(\alpha z), \quad n \geq 1$$

with $x_0(t) = Ae^{-\lambda t}$.

Hence show that for small values of ε the approximate solution is

$$x(t) = Ae^{-\lambda t} + \frac{\varepsilon A}{\lambda(1-\alpha)}\left(e^{-\lambda\alpha t} - e^{-\lambda t}\right)$$
$$+ \frac{\varepsilon^2 A}{\lambda^2(1-\alpha)^2}\left\{e^{-\lambda t} - e^{-\lambda\alpha t} - \frac{1}{1+\alpha}\left(e^{-\lambda t} - e^{-\lambda\alpha^2 t}\right)\right\}.$$ [12]

(c) Consider the Hamiltonian

$$H(q,p) = \tfrac{1}{2}p^2 + \tfrac{1}{2}q^2 + \varepsilon \exp(q^2)$$

where ε is a small constant. Show that, to first order in ε, the Hamiltonian in the angle-action variables, (ϕ, J), of H can be written in the form

$$H(J) = J + \varepsilon\left(1 + J + \tfrac{3}{4}J^2 + \cdots\right)$$

and hence find, to first order in ε, an approximation to the frequency of the perturbed motion as a function of J. [8]

Hint: you may assume that the angle-action variables, (θ, I), of the unperturbed system, $\varepsilon = 0$, are given by

$$q = \sqrt{2I}\sin\theta, \quad p = \sqrt{2I}\cos\theta,$$

and that

$$\exp x = 1 + x + \tfrac{1}{2}x^2 + \cdots.$$