PART I

You should answer at least one question from this part.

Question 1

(a) The function $v(x)$ has three zeros $x_1 < x_2 < x_3$, with x_1 and x_2 simple zeros, but with $v'(x_3) = 0$; for $x > x_3$, $v(x) > 0$. Provide a qualitative description of the motion of the system with velocity function $v(x)$, being careful to discuss the nature of each fixed point.

If ε is an arbitrarily small positive number describe the qualitative differences between the phase diagrams of the two systems

$$\dot{x} = v(x)^2 + \varepsilon, \quad \text{and} \quad \dot{x} = v(x)^2 - \varepsilon,$$

where $v(x)$ is the function described above. [13]

(b) Consider the linear second-order system $\dot{x} = Ax$, where the matrix A is given by

$$A = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}.$$

Classify the fixed point of this linear system, and make a rough sketch showing the qualitative features of the motion in the neighbourhood of the origin. [7]

(c) A linear second-order system $\dot{x} = Bx$, where B is a real, constant, non-singular 2×2 matrix, is perturbed by adding to its velocity field a time-independent term of order $|x|^2$, so that the equation of motion becomes

$$\dot{x} = Bx + O(|x|^2).$$

Under what conditions can the qualitative behaviour of the nonlinear system near the origin be obtained from the properties of B? [5]

Question 2

(a) Sketch the graph of the function

$$V(q) = -\frac{q^2}{1+q^4}.$$

Show that the Hamiltonian

$$H(q,p) = \tfrac{1}{2}p^2 - \frac{q^2}{1+q^4},$$

has two stable fixed points. Find and classify any other fixed points. [8]

(b) Sketch some representative contours of this Hamiltonian, being careful to include all invariant regions and any separatrixes. Give the range of energies for which bound motion is possible. [9]

(c) Show that in the vicinity of either stable fixed point, q_f, the Hamiltonian has the form

$$H(x,p) = \tfrac{1}{2}p^2 + x^2 - \tfrac{1}{2} + O(x^3), \quad x = q - q_f.$$

Hence, or otherwise, find an approximate expression for $q(t)$ in the neighbourhood of q_f. [5]

(d) Describe, qualitatively, the variation of the period of the motion around either stable fixed point as the energy increases towards the separatrix energy. [3]