PART II

You should answer at least one question from this part.

Question 5

(a) Use perturbation theory to show that an approximate solution to the equation

$$\sin x = \varepsilon e^{-\alpha x}$$

where ε is a small positive parameter and α a positive constant, is

$$x = \varepsilon - \alpha\varepsilon^2 + \left(\frac{1}{6} + \frac{3\alpha^2}{2}\right)\varepsilon^3 + O(\varepsilon^4).$$ [5]

(b) Show that, to first order in ε, the solution of the differential equation

$$\dot{x} = \frac{1}{x} + \varepsilon e^{-x}, \quad x(0) = A > 0,$$

can be written in the form

$$x(t) = w(t) + \varepsilon y(t), \quad w = \sqrt{A^2 + 2t}$$

where $y(w)$ satisfies the equation

$$w\frac{dy}{dw} + y = w^2 e^{-w}, \quad y(A) = 0.$$

Hence show that an approximate solution to the original equation is

$$x(t) = w + \frac{\varepsilon}{w}\int_A^w dz\, z^2 e^{-z} \quad \text{with} \quad w(t) = \sqrt{A^2 + 2t}.$$ [12]

(c) Consider the Hamiltonian

$$H(q,p) = \tfrac{1}{2}p^2 + \tfrac{1}{2}q^2 + \frac{\varepsilon}{(1+pq)^2}, \quad |pq| < 1,$$

where ε is a small constant. Show that, to first order in ε, the Hamiltonian in the angle-action variables, (ϕ, J), of H is

$$H(J) = J + \frac{\varepsilon}{(1-J^2)^{3/2}}, \quad 0 \le J < 1,$$

and hence find a first order approximation to the frequency of the perturbed motion as a function of J. [8]

Hint: you may assume that the angle-action variables, (θ, I), of the unperturbed system, $\varepsilon = 0$, are given by

$$q = \sqrt{2I}\sin\theta, \quad p = \sqrt{2I}\cos\theta$$

and that

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{dx}{(1+a\sin x)^2} = \frac{1}{(1-a^2)^{3/2}}, \quad |a| < 1.$$