PART I

You should answer at least one question from this part.

Question 1

(a) The function $v(x)$ has three zeros $x_1 < x_2 < x_3$ with x_1 and x_2 simple zeros, but with $v'(x_3) = 0$; for $x > x_3$, $v(x) > 0$. Provide a qualitative description of the motion of the system with velocity function $v(x)$, being careful to discuss the nature of each fixed point.

If ε is an arbitrarily small positive number describe the qualitative differences between the phase diagrams of the two systems

$$\dot{x} = v(x) + \varepsilon, \quad \text{and} \quad \dot{x} = v(x) - \varepsilon,$$ [12]

where $v(x)$ is the function described above.

(b) Consider the linear second-order system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where the matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

Classify the fixed point of this linear system, and make a rough sketch showing the qualitative features of the motion in the neighbourhood of the origin. [8]

(c) A linear second-order system $\dot{\mathbf{x}} = \mathbf{B}\mathbf{x}$, where $\mathbf{B}$ is a real, constant, non-singular 2×2 matrix, is perturbed by adding to its velocity field a time-independent term of order $|\mathbf{x}|^2$, so that the equation of motion becomes

$$\dot{\mathbf{x}} = \mathbf{B}\mathbf{x} + O(|\mathbf{x}|^2).$$

Describe the conditions that the matrix $\mathbf{B}$ must satisfy in order that, near the origin, the phase curves of the non-linear and linear systems are qualitatively similar. [5]

Question 2

(a) Sketch the graph of the function

$$V(q) = q^4 - q^2.$$

Locate and classify the fixed points of the Hamiltonian

$$H(q,p) = \tfrac{1}{2}p^2 + q^4 - q^2,$$

and sketch some representative contours of it, including any separatrixes. [14]

(b) Show that in the vicinity of a stable fixed point, q_f, the Hamiltonian has the form

$$H(x,p) = \tfrac{1}{2}p^2 + 2x^2 - \tfrac{1}{4} + O(x^3), \quad x = q - q_f.$$

Hence, or otherwise, find an approximation to the period of the motion in the vicinity of this stable fixed point. [6]

(c) Write down the equation for $q(t)$, valid near q_f, for the initial conditions $q(0) = q_f + \delta$, $\dot{q}(0) = 0$, where δ is a small number. [5]