Question 8

(a) Show that the two-dimensional map $(x_n, y_n) \to (x_{n+1}, y_{n+1})$

$$\mathbf{T}: \begin{array}{l} x_{n+1} = 2f(x_n) - y_n \\ y_{n+1} = x_n \end{array}$$

where $f(x)$ is an arbitrary, differentiable function, is area-preserving. [4]

(b) Show that the only fixed points of $\mathbf{T}$ are at $(x, y) = (x_f, x_f)$ where x_f is a root of the equation

$$x_f = f(x_f).$$

Further, show that these fixed points are stable or unstable according as

$$|f'(x_f)| < 1 \quad \text{or} \quad |f'(x_f)| > 1.$$ [4]

(c) Show that the fixed points of $\mathbf{T}^2$ are at $(x, y) = (u, v)$ where u is a root of the equation

$$u = f(f(u)) \quad \text{and} \quad v = f(u).$$ [4]

(d) In the particular case $f(x) = cx - x^2$, where c is a positive constant, show that the only fixed points of $\mathbf{T}$ are at $(0,0)$ and $(c-1, c-1)$ and that for $0 < c < 1$ the fixed point at $(0,0)$ is stable, but is otherwise unstable.

Determine the values of c for which the fixed point at $(c-1, c-1)$ is stable and for which it is unstable. [5]

(e) Show that the fixed points, (u, v), of $\mathbf{T}^2$ are at the roots of

$$u(c-u)(u^2 - cu + c) = u \quad \text{and} \quad v = f(u).$$

Show that two of these roots are at $u = 0$ and $u = c - 1$ and by putting $w = u - c + 1$ show that the remaining fixed points are given by the roots of

$$w^2 + (c-3)w - (c-3) = 0$$

and that this gives

$$u = \tfrac{1}{2}\sqrt{c+1}\left(\sqrt{c+1} \pm \sqrt{c-3}\right), \quad c \geq 3.$$

Describe, in qualitative terms, the difference between the orbits in the neighbourhood of the origin for the case $c = 3 - \delta$ and $c = 3 + \delta$, where δ is a small positive number. [8]

[END OF QUESTION PAPER]