Question 6

Consider the Hamiltonian of a particle of mass m,

$$H = \frac{1}{2m}p^2 + V(q) + f(t)V_1(q)$$

where $f(t)$ is a slowly varying function of the time; it may be assumed that if f is held constant at a value within its range of variation then the motion is bound.

(a) Write down the integral expression for the action variable, I, in the limit that f is a constant. Describe, qualitatively, the behaviour of $I(t)$ if f is allowed to vary slowly. [5]

(b) By using the principle of adiabatic invariance, and by differentiating under the integral sign, show that if $f(t)$ varies sufficiently slowly, then the rate of change of the energy is given by

$$\frac{dE}{dt} = M(E,t)\frac{df}{dt},$$

where M is the mean value of $V_1(q)$ evaluated over one period of an orbit computed as if f were constant,

$$M(E,t) = \frac{1}{T(E,t)}\int_0^T dt\, V_1(q(E,t)),$$

where T is the period, again computed as if f were held constant. [12]

(c) For the Hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\left(\omega^2 + \alpha^2 f(t)\right)q^2,$$

use the result obtained in part (b) to show that,

$$\frac{E(t)}{E(0)} = \sqrt{\frac{\omega^2 + \alpha^2 f(t)}{\omega^2 + \alpha^2 f(0)}}.$$ [8]

Question 7

A particle of mass m is moving subject to a rapidly varying force, the Hamiltonian being

$$H(q,p,t) = \frac{p^2}{2m} + \frac{Fq^2}{1+q^2}\sin\Omega t$$

where F is a constant, and the frequency Ω is large.

(a) Show that the mean motion Hamiltonian is given in terms of the mean motion coordinates (Q,P) by

$$\overline{K}(Q,P,t) = \frac{P^2}{2m} + \frac{F^2}{m\Omega^2}\frac{Q^2}{(1+Q^2)^4}.$$

Give a brief account of the approximations used in deriving this Hamiltonian. [8]

(b) Provide a sketch of the effective potential as a function of Q and hence give a brief qualitative description of the motion in phase space. [10]

(c) Show that a particle moving towards the origin, but starting a long way away from it, will be turned back if its initial speed is less than

$$\frac{F}{m\Omega}\sqrt{\frac{3^3}{2^7}}.$$ [7]