Question 4

(a) Sketch the phase curves for a particle of mass m moving in the potential

$$V(q) = \begin{cases} \infty, & q < 0 \\ aq^{1/3}, & q > 0 \end{cases}$$

where a is a positive constant. [4]

(b) Show that for the system with Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + V(q),$$

where $V(q)$ is the potential defined in part (a), the action variable I is given in terms of the energy E by

$$I = \frac{6E^{7/2}\sqrt{2m}}{\pi a^3} \int_0^{\pi/2} d\phi \, \sin^5\phi \cos^2\phi,$$

where $q = (E/a)^3 \sin^6\phi$, and deduce that

$$E = \left(\frac{35\pi a^3}{16\sqrt{2m}} \right)^{2/7} I^{2/7}.$$

Hence find an expression for the frequency of the motion as a function of the energy. [12]

(c) Show that the angle variable θ is given by

$$\theta(\phi) = \pi \left(1 - \tfrac{75}{64}\cos\phi + \tfrac{25}{128}\cos 3\phi - \tfrac{3}{128}\cos 5\phi\right), \quad 0 \le \phi \le \tfrac{\pi}{2}$$

where $\theta = 0$ when $q = 0$. [5]

(d) If a is allowed to vary slowly from a_1 to a_2 during a time long by comparison with the period, give an approximate expression for the ratio of the frequencies of the motion before and after this change has occurred. [4]

You may assume the results

$$\int_0^{\frac{\pi}{2}} dx \, \sin^{2n+1} x = \frac{(2n).(2n-2)\ldots 2}{(2n+1).(2n-1)\ldots 3.1},$$

where n is any positive integer, and

$$\sin^5 x = \tfrac{1}{16}(10\sin x - 5\sin 3x + \sin 5x).$$