M336/A

The Open University

Third Level Course Examination 1998
Groups and Geometry

Monday, 12th October, 1998 10.00 am – 1.00 pm

Time allowed: 3 hours

There are **TWO** parts to this examination. You should attempt **BOTH**.

You may attempt **ALL** the questions in Part I. You should answer not more than **THREE** questions from Part II, including not more than **TWO** questions from Part IIA and not more than **TWO** questions from Part IIB.

You are advised to spend about 90 minutes on Part I and 80 minutes on Part II, and to leave yourself about 10 minutes for checking. Part I carries 55% of the total marks while Part II carries 45% of the total marks.

You are advised to show all your working and to give reasons for all your answers unless a question is explicitly phrased 'Write down...'. You should begin each answer on a new page of the answer book.

Note: a Figure Sheet is provided with this question paper for use in answering Questions 7, 11 and 13.

At the end of the examination

Check that you have written your examination number and personal identifier on each answer book used and on any separate sheets used. **Failure to do so may mean that your paper cannot be identified.** Attach all your answer books and the Figure Sheet together, using the fastener provided.

Write the numbers of the questions that you have attempted in the spaces provided on the front of the answer book.

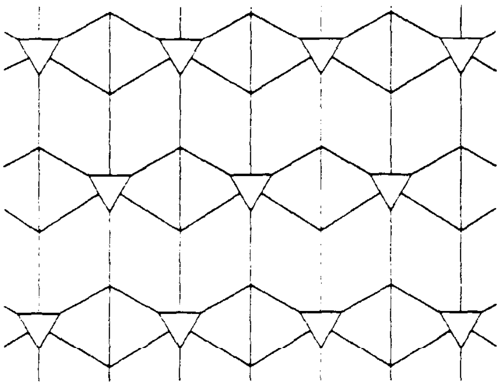
Part I

You may attempt **ALL** *the questions in this part and are advised to spend about* **90 minutes** *on it.*

The marks for each question are given beside the question.

Question 1

Consider the following tiling, $\mathcal{T}$.



$\mathcal{T}$

(a) Write down all the tile types and vertex types in $\mathcal{T}$. [3]

(b) Is $\mathcal{T}$ edge-to-edge? [1]

(c) Which one of the vertex types corresponds to two distinct vertex orbits under the symmetry group of $\mathcal{T}$? [1]

Question 2

The group G is given by the presentation

$$G = \langle r, s : r^4 = e, r^2 = s^2, sr = r^3 s \rangle$$

and the elements are written in standard form as

$$r^m \text{ and } r^m s, \qquad m = 0, \ldots, 3.$$

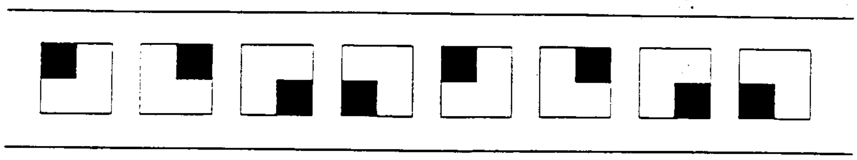
Note that G is *not* D_4.

(a) Express the element $(r^2 s)^2$ in standard form. [3]

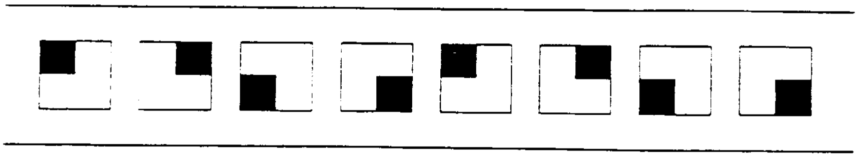
(b) Prove that the element $r^2 s$ has order 4. [2]

Question 3

Use the algorithm of *Unit IB3* to determine the types of the following friezes, $\mathcal{F}$ and $\mathcal{G}$.



$\mathcal{F}$



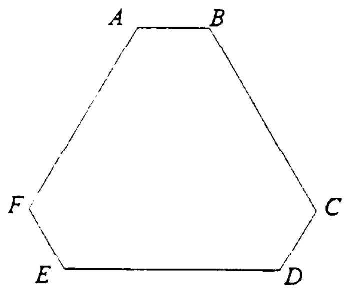
$\mathcal{G}$ [5]

Question 4

Let G be a group, H a subgroup of G and N a normal subgroup of G.
Prove that $H \cap N$ is a normal subgroup of H. [5]

Question 5

Consider the following hexagon, with three long and three short sides, whose symmetry group G is of order 6.



(a) Write down the cycle index of G, considered as acting on the six corners A, B, C, D, E, F. [2]

(b) Write down the cycle index of G, considered as acting on the six sides AB, BC, CD, DE, EF, FA. [2]

(c) The sides of the hexagon can be coloured red or blue. Use the cycle index in part (b) to determine the number of equivalence classes of such configurations. [1]

Question 6

(a) Write down all the *subgroups* of order 3 of the direct product

$$\mathbb{Z}_{45} \times \mathbb{Z}_9.$$

You should express each subgroup *exactly once* as $\langle(a, b)\rangle$ for a suitable element of $\mathbb{Z}_{45} \times \mathbb{Z}_9$. [4]

(b) How many *elements* of order 3 does $\mathbb{Z}_{45} \times \mathbb{Z}_9$ have? [1]

Question 7

Let $\mathcal{T}$ be the tiling considered in Question 1, an enlarged portion of which is provided as Figure 1 of the Figure Sheet.

(a) On Figure 1 of the Figure Sheet, indicate the translational tile orbits by placing numbers on the tiles. [2]

(b) Write down $n_v(\mathcal{T})$. [1]

(c) Using a theorem from *Unit GE2*, or otherwise, find $n_e(\mathcal{T})$. [2]

Question 8

The finitely presented Abelian group A is defined by

$$A = \langle a, b, c : 2a + 2c = 0, 6b + 6c = 0, 2a + 12b + 14c = 0 \rangle.$$

(a) Write down the integer matrix representing A. [1]

(b) Use the Reduction Algorithm to reduce the matrix to diagonal form. [3]

(c) Is the group A finite or infinite? Justify your answer. [1]

Question 9

Let $\mathbf{a} = (2, 0)$, $\mathbf{b} = (2, 1)$ and $\mathbf{c} = (2, -1)$.

(a) Write down a reduced basis for $L(\mathbf{a}, \mathbf{b})$. [1]

(b) Write down the lattice type of $L(\mathbf{a}, \mathbf{b})$. [1]

(c) Write down the lattice type of $L(\mathbf{b}, \mathbf{c})$. [1]

(d) Using any convenient notation, write down a glide reflection belonging to $\Gamma(L(\mathbf{b}, \mathbf{c}))$, that maps $\mathbf{0}$ to $\mathbf{b}$. [2]

Question 10

The class equation of the permutation group S_4 is

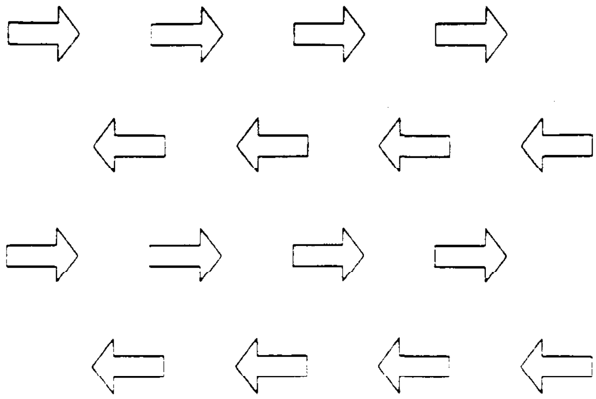
$$24 = 1 + 3 + 6 + 6 + 8.$$

(a) Write down the elements of the conjugacy class corresponding to '3' in the class equation. [1]

(b) Prove that the union of this conjugacy class with the subset consisting of the identity element is a normal subgroup of S_4. [4]

Question 11

Consider the wallpaper pattern W below, a copy of which is reproduced as Figure 2 on the Figure Sheet.



W

(a) On Figure 2 of the Figure Sheet:

(i) draw a basic rectangle of W;

(ii) within your basic rectangle, shade a generating region. [3]

(b) Write down the wallpaper type of W. (You need not give a justification.) [2]

Part II

You may attempt not more than **THREE** *questions from this part, and you are advised to spend about* **80 minutes** *on it.*

You may choose not more than **TWO** *of your three questions from Part IIA and not more than* **TWO** *from Part IIB.*

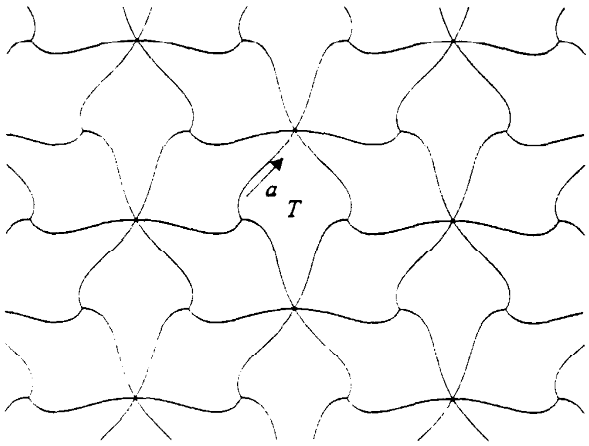
Each question carries 15% of the marks for the whole examination, and an indication of the allocation of marks within each question is given beside the question.

Part IIA

Do not attempt more than two questions from Part IIA.

Question 12

Consider the following transitive tiling $\mathcal{T}$, a copy of which is reproduced as Figure 3 on the Figure Sheet.



$\mathcal{T}$

(a) Starting at the marked edge of tile T in Figure 3 on the sheet, fill in appropriate edge side labels on both the inside and the outside of each edge of T. [3]

(b) Hence write down the incidence symbol of $\mathcal{T}$. [2]

(c) How many *edge side* orbits are there under the action of $\Gamma(\mathcal{T})$? [1]

(d) How many *edge* orbits are there under the action of $\Gamma(\mathcal{T})$? [1]

(e) Write down the number, n_3, of translational orbits of vertices of degree 3, and also the number, n_6, of translational orbits of vertices of degree 6. [1]

(f) Hence find the number of translational orbits of edges. [2]

(g) *Neither* of the following can be the incidence symbol of a transitive tiling. For each one, explain briefly why not.

(i) $\vec{a}\, b\, \vec{c}$

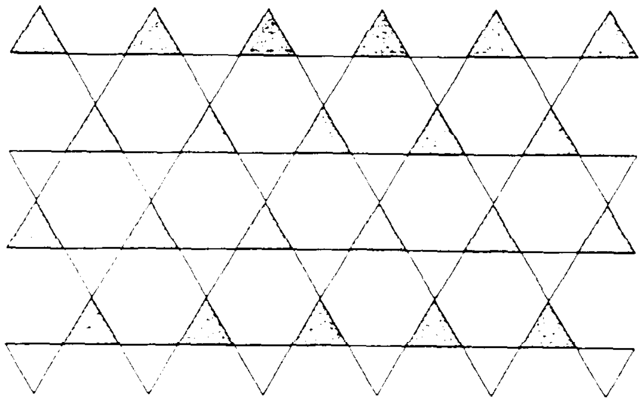
$\vec{a}\, b\, \vec{c}$

(ii) $\vec{a}\,\vec{b}\,\vec{a}\,\vec{c}$

$\overleftarrow{a}\,\overleftarrow{c}\,\overleftarrow{a}\,\overleftarrow{b}$ [5]

Question 13

Consider the following wallpaper pattern W, which has been formed from the Archimedean tiling (3,6,3,6) by shading some of the triangular tiles. An enlarged copy of W is reproduced as Figure 4 on the Figure Sheet.



W

(a) How many orbits of rotation centres are there under the action of the *symmetry* group $\Gamma(W)$? Clearly mark *one* representative of each orbit, using Figure 4 of the Figure Sheet. [3]

(b) Carry out the algorithm described in the audio tape for *Unit GE4*, to determine the type of W. You should explain at each stage your answer to the question posed by the algorithm. [5]

(c) On Figure 4 of the Figure Sheet, draw an equilateral triangle which is also a generating region for W. [2]

(d) Assume that the original tiling is by tiles whose sides are of unit length. By comparing the area of your generating region with that of a fundamental parallelogram, or otherwise, find the length of a side of the generating region. [5]

Question 14

(a) For each of the following Bravais lattices $L(\mathbf{a},\mathbf{b},\mathbf{c})$, find the offset of $\mathbf{c}$ with respect to $\mathbf{a}.\mathbf{b}$ and then find the type of the lattice.

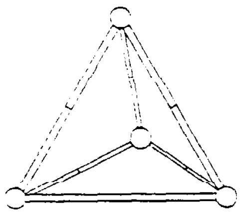
(i) $\mathbf{a} = (1,0,0)$, $\mathbf{b} = (0,3,0)$, $\mathbf{c} = \left(\frac{1}{2},\frac{3}{2},2\right)$.

(ii) $\mathbf{a} = \left(1,\sqrt{3},0\right)$, $\mathbf{b} = \left(-1,\sqrt{3},0\right)$, $\mathbf{c} = \left(0,2\sqrt{3},1\right)$.

(iii) $\mathbf{a} = (1,0,0)$, $\mathbf{b} = (0,1,0)$, $\mathbf{c} = \left(\sqrt{5},\sqrt{6},\sqrt{7}\right)$.

[9]

(b) A supply of identical rods is available, each 10 cm long, which can be joined in pairs to produce rods 20 cm long. There is also a supply of spherical joints which can be used as vertices, to form the rods into tetrahedral frameworks: see below.



(i) Let G be the *direct* symmetry group of such a framework. Write down the cycle index of G, considered as acting *on the twelve 10 cm rods*, that is, on the 'half-edges' of the tetrahedron. [2]

(ii) Suppose now that each 10 cm rod (that is, each 'half-edge') may be painted either yellow or blue. Calculate how many rotational classes of such colourings there are. [2]

(iii) Given that the number of equivalence classes of colourings under the *full* symmetry group of the framework is 218, determine how many of the rotational classes of colourings are indistinguishable from their mirror images. [2]

Part IIB

Do not attempt more than two questions from Part IIB.

Question 15

Let H be a subgroup of a group G. The *normalizer*, L, of H in G is defined by

$$L = \{x \in G : xH = Hx\}.$$

(a) Prove that L is a subgroup of G and that H is a normal subgroup of L. [7]

(b) Prove that if K is any subgroup of G containing H such that H is a normal subgroup of K, then K is a subgroup of L. [3]

(c) Find the normalizers of the following two subgroups of the permutation group S_3:

(i) $\{e, (1\,2)\}$;

(ii) $\{e, (1\,2\,3), (1\,3\,2)\}$.

(Give a brief reason for each answer.) [5]

Question 16

(a) Find all Abelian groups of order 1575 (up to isomorphism), expressing each of them *both* in canonical form *and* in p-primary form. [9]

(b) Write down the groups that you found in part (a) which possess a subgroup isomorphic to $\mathbb{Z}_{45}$, justifying your answer. [6]

Question 17

Let G be a group of order 72.

(a) Show that G has either 1 or 4 Sylow 3-subgroups. [3]

(b) Consider the case where G has 4 Sylow 3-subgroups X_1, X_2, X_3 and X_4.

Using the alternative definition of group action given in Lemma 5.1 of *Unit IB2*, show that conjugation by elements of G on the elements of the set

$$X = \{X_1, X_2, X_3, X_4\}$$

produces an action of the group G on the set X and hence a homomorphism ϕ from G to the permutation group S_4. [6]

(c) Prove that kernel of the homomorphism ϕ of part (b) is neither G nor $\{e\}$. [4]

(d) Use parts (a) and (c) to show that no group of order 72 is simple. [2]

[END OF QUESTION PAPER]

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Figure Sheet

Answer Sheet
for Questions
7, 11, 12 and 13

Please complete the identification grid below.

Examination No.	
Personal Identifier	

Question 7
Please read carefully the instructions given for Question 7 of the question paper

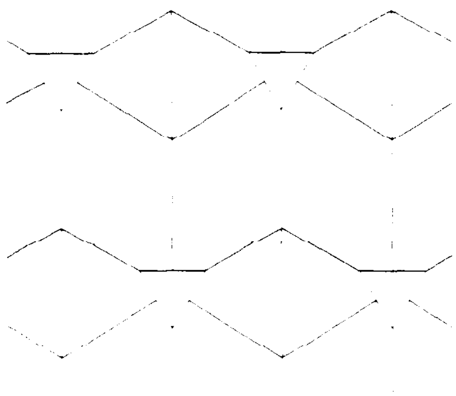

Figure 1

Question 11
Please read carefully the instructions given for Question 11 of the question paper

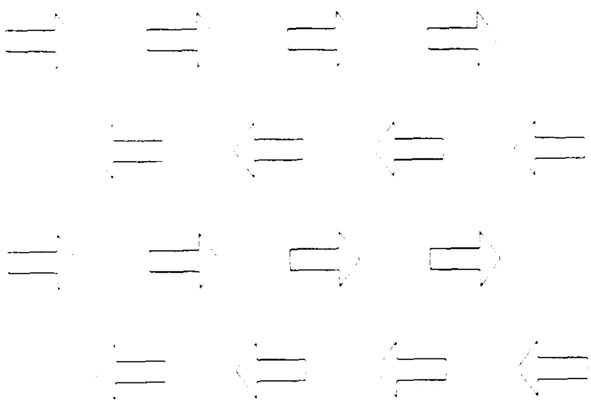

Figure 2

For examiner's use	

Question 12
Please read carefully the instructions given for Question 12 of the question paper

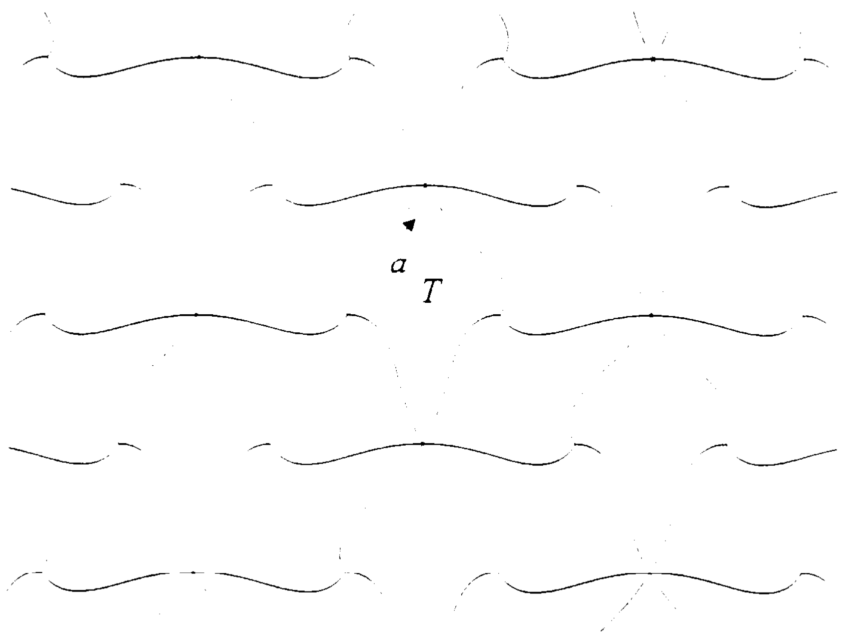

Figure 3

Question 13
Please read carefully the instructions given for Question 13 of the question paper

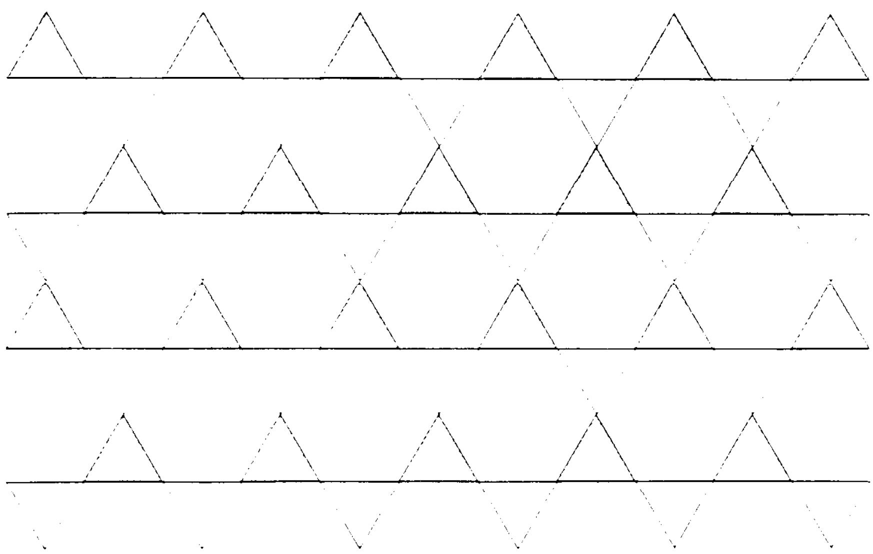

Figure 4

For examiner's use	