M336/F

The Open University

Third Level Course Examination 1997

Groups and Geometry

Wednesday, 15th October, 1997 2.30pm–5.30pm

Time allowed: 3 hours

There are **TWO** parts to this examination. You should attempt **BOTH**.

You may attempt **ALL** the questions in Part I. You should answer not more than **THREE** questions from Part II, including not more than **TWO** questions from Part IIA and not more than **TWO** questions from Part IIB.

You are advised to spend about 90 minutes on Part I and 80 minutes on Part II, and to leave yourself about 10 minutes for checking. Part I carries 55% of the total marks while Part II carries 45% of the total marks.

You are advised to show all your working and to give reasons for all your answers unless a question is explicitly phrased 'Write down...'. You should begin each answer on a new page of the answer book.

Tracing paper is available from the invigilator if you should require it.

Note: a Figure Sheet is provided with this question paper for use in answering Questions 7 and 13

At the end of the examination

Check that you have written your name, examination number and personal identifier on each answer book used and on any separate sheets used, *particularly the Figure Sheet.* **Failure to do so may mean that your paper cannot be identified.** Attach all your answer books and the Figure Sheet together, using the fastener provided.

Write the numbers of the questions that you have attempted in the spaces provided on the front of the answer book.

Part I

You may attempt **ALL** *the questions in this part and are advised to spend about* **90 minutes** *on it.*

The marks for each question are given beside the question.

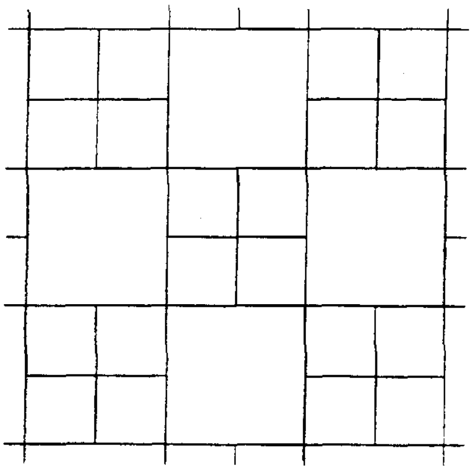
Question 1

(a) Let $f = t[\mathbf{p}]\lambda[\mathbf{A}]$, where $\mathbf{p} = (1, 2)$ and

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}.$$

Find f^2. [3]

(b) Write down all the tile types and vertex types in the following tiling.



[2]

Question 2

The group G is given by the presentation

$$G = \langle r, s : r^6 = e, r^3 = s^2, sr = r^5 s \rangle$$

and the elements are written in standard form as

$$r^m \text{ and } r^m s, \qquad m = 0, \ldots, 5.$$

Note that G is *not* D_6.

(a) Express the product

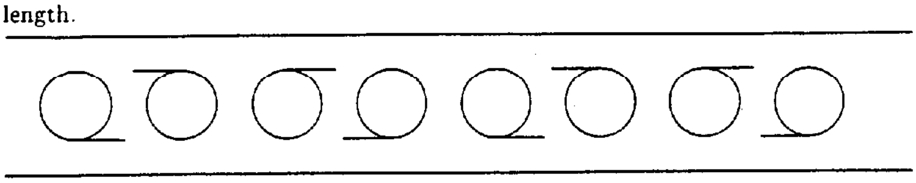
$$(r^4 s)(r^3 s)$$

in standard form. [2]

(b) Express $(r^4 s)^{-1}$ in standard form. [3]

Question 3

(a) In the following frieze, $\mathcal{F}$, the circles are equally sized and spaced, with their centres on the centre line of the frieze. The horizontal 'tails' are all of equal length.



Use the algorithm of *Unit IB3* to determine the type of $\mathcal{F}$, making clear which questions you ask and what the corresponding answers are. [4]

(b) If each rightward pointing 'tail' were removed and each leftward pointing 'tail' retained, write down the frieze type that would result. (You need not give a justification for your answer.) [1]

Question 4

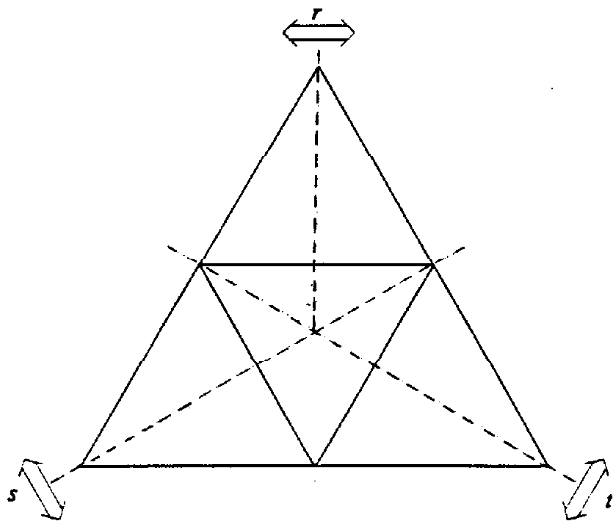
Let G be a group, H a subgroup of G and N a normal subgroup of G. Prove that the set HN, defined by

$$HN = \{hn : h \in H, n \in N\},$$

is a subgroup of G. [5]

Question 5

Four equilateral triangles made of coloured glass are stuck together along edges to form a larger equilateral triangle:



The symmetry group of the resulting object is

$$G = \{e, f, g, r, s, t\},$$

where f is anticlockwise rotation through $2\pi/3$, $g = f^2$, and r, s and t are the reflections indicated in the diagram.

(a) Write down the cycle index of G, considered as acting on the four small glass triangles. [2]

(b) Each small triangle can be blue (B) or yellow (Y). Write down the pattern inventory for the configurations that can appear. (You need *not* expand the expression.) [2]

(c) How many equivalence classes of such configurations are there? [1]

Question 6

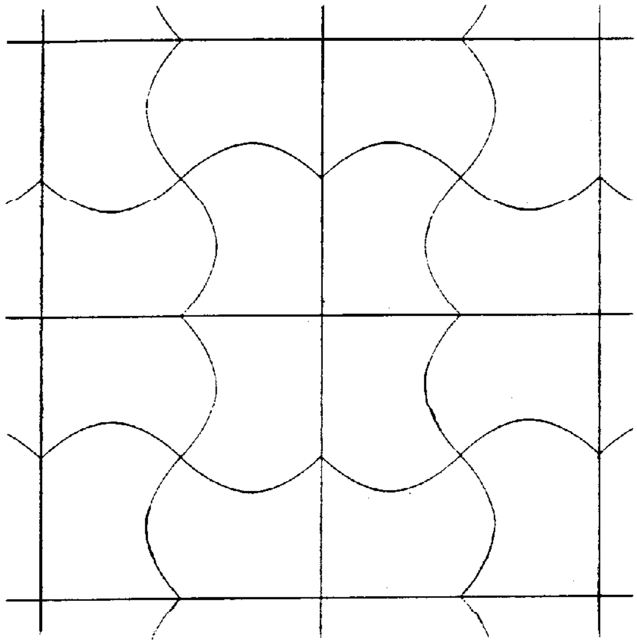
(a) Use the Euclidean Algorithm to find hcf$\{138, 102\}$. [2]

N.B. You *must* use the algorithm; *no marks* will be awarded for factorizing the two numbers.

(b) Use your answer to part (a) to express hcf$\{138, 102\}$ as an integer combination of 138 and 102. [3]

Question 7

Consider the following transitive tiling $\mathcal{T}$.



$\mathcal{T}$

(a) Using the enlarged drawing of one tile T of $\mathcal{T}$ supplied as Figure 1 of the Figure Sheet, start at a straight edge and produce the edge side labels on both sides of each edge of T. [2]

(b) Write down the incidence symbol of $\mathcal{T}$. [1]

(c) Write down the edge stabilizer of a straight edge of $\mathcal{T}$. [1]

(d) Write down the edge stabilizer of a curved edge of $\mathcal{T}$. [1]

Question 8

The finitely presented Abelian group A is defined by

$$A = \langle a, b, c : 2a + 4b + 6c = 0, 10a + 22b + 48c = 0, 6a + 20b + 48c = 0 \rangle.$$

(a) Write down the matrix representing A. [1]

(b) Use the Reduction Algorithm to reduce the matrix to diagonal form. [3]

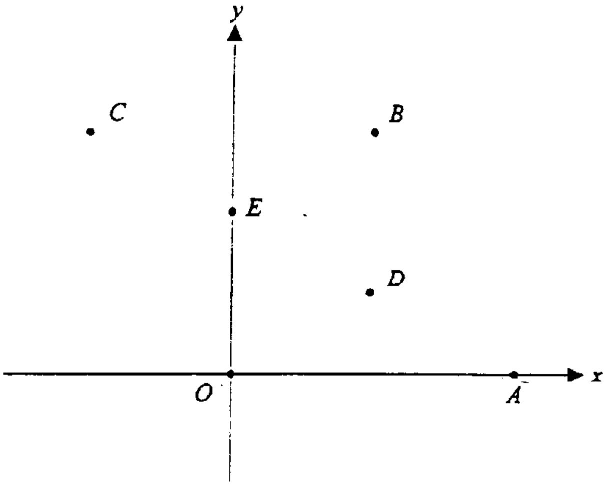
(c) Hence express A as a direct product of non-trivial cyclic groups in canonical form. [1]

Question 9

Let L be the plane hexagonal lattice $L(\mathbf{a}, \mathbf{b})$, where

$$\mathbf{a} = (1, 0), \quad \mathbf{b} = (1/2, \sqrt{3}/2).$$

Let O, A, B and C be the points in the plane whose vectors are $\mathbf{0}$, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{b} - \mathbf{a}$, respectively. (Note that OAB is an equilateral triangle.)



(a) Write down (as linear combinations of a and b):

 (i) the vector of the centroid D of triangle OAB,

 (ii) the vector of the centroid E of triangle OBC. [2]

(b) Write down, using any convenient notation, the reflection in $\Gamma(L)$ that maps E to D. [1]

(c) Write down, using any convenient notation, a glide reflection in $\Gamma(L)$ that maps E to D. [2]

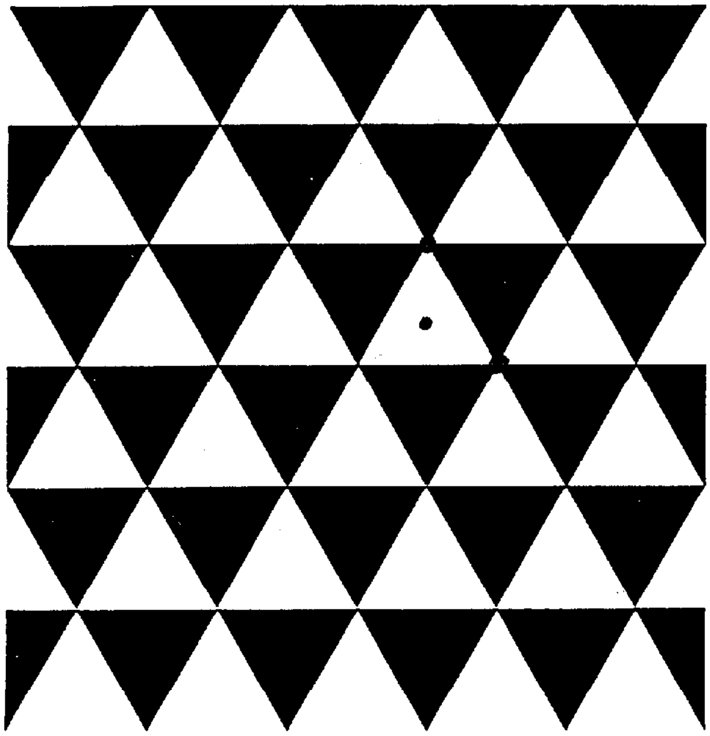
Question 10

(a) How many Abelian groups are there of order 108? Justify your answer. [2]

(b) Write down all the Abelian groups of order 108 which have one or more cyclic subgroups of order 9. [3]

Question 11

Carry out the algorithm described in the Handbook to determine the type of the following wallpaper pattern, W, based on black and white equilateral triangles:



W

Your solution should make it clear which questions you ask and what the corresponding answers are. [5]

Part II

You may attempt not more than **THREE** *questions from this part, and you are advised to spend about* **80 minutes** *on it.*

You may choose not more than **TWO** *of your three questions from Part IIA and not more than* **TWO** *from Part IIB.*

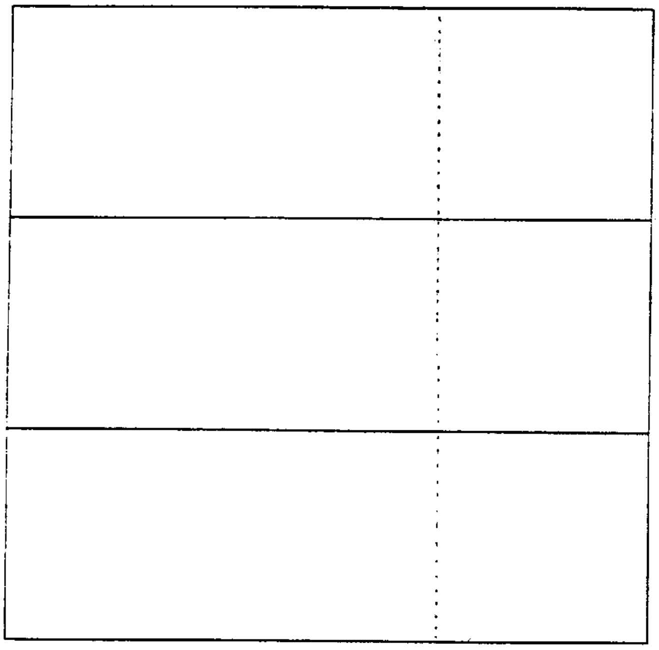
Each question carries 15% of the marks for the whole examination, and an indication of the allocation of marks within each question is given beside the question.

Part IIA

Do not attempt more than two questions from Part IIA.

Question 12

Each face of a square card is divided into three congruent rectangles, the dividing lines on one face being at right angles to those on the other.



Dotted lines are on the underside of the card.

(a) Let G be the group of rotational symmetries (some of which involve turning the card over) of the above configuration of six rectangles, three on each side of the card. Describe these symmetries geometrically. [4]

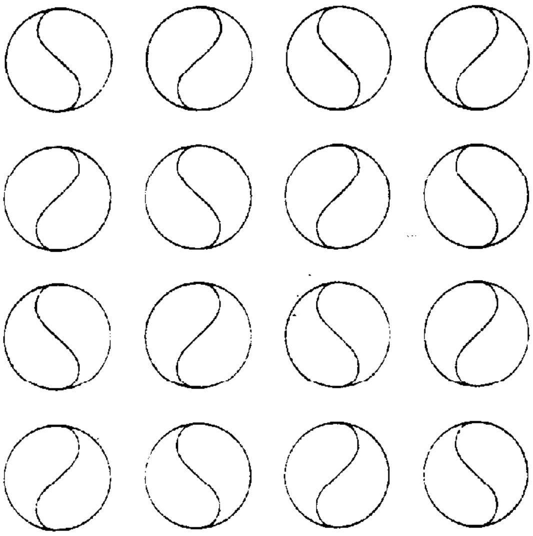
(b) Find the cycle index of G acting on this configuration. [4]

(c) The six rectangles are to be coloured, each rectangle being one of *three* colours. Calculate the number of equivalence classes of such colourings. [3]

(d) When a group G acts on a set X, the *inert pairs* of the action are the ordered pairs (x, g) such that $x \in X$, $g \in G$ and $g \wedge x = x$. Explain briefly how this concept can be used to prove the Counting Lemma. [4]

Question 13

Consider the following wallpaper pattern W below, an enlarged copy of which is reproduced as Figure 2 on the Figure Sheet.



W

(a) Carry out the algorithm described in the Handbook to determine the type of the pattern W. Your solution should make it clear which questions you ask and what the corresponding answers are. [5]

(b) Using Figure 2 on the Figure Sheet, indicate with a clearly visible cross or dot *one* representative of *each* orbit under $\Gamma(W)$ of non-trivial rotation centres. [2]

(c) On Figure 2 of the Figure Sheet, draw *two successive* representatives of *each* orbit under $\Gamma(W)$ of axes of indirect symmetry. (Use solid lines for any reflection axes and/or dashed or dotted lines for any glide axes.) [4]

(d) In Figure 3 of the Figure Sheet, the shading of W has been removed. Shade *either* the left *or* the right part of some or all of the motifs in such a way as to produce a type of wallpaper pattern with no indirect symmetries but *with* non-trivial rotation symmetries. (Alternatively, you may describe *very clearly* how to shade the pattern to achieve the desired type.) [4]

Question 14

(a) Let $L = L(\mathbf{a}, \mathbf{b}, \mathbf{c})$ be a Bravais lattice, where

$$\mathbf{a} = (3, 1, 0), \quad \mathbf{b} = (0, 0, 2), \quad \mathbf{c} = (0, 2, 0).$$

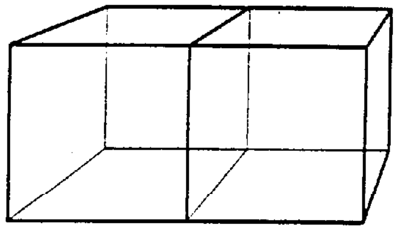
(i) Explain how Theorem 1.2 (page 55 of the Handbook) may be used to express L as $L(\mathbf{a}', \mathbf{b}', \mathbf{c}')$, where $L(\mathbf{a}', \mathbf{b}')$ is a square lattice. [3]

(ii) Use the result of part (a)(i) to find the type of L. [3]

(b) Find the Bravais lattice type of $L(\mathbf{d}, \mathbf{e}, \mathbf{f})$, where

$$\mathbf{d} = (\sqrt{2}, \sqrt{2}, 2), \quad \mathbf{e} = (-\sqrt{2}, -\sqrt{2}, 2), \quad \mathbf{f} = (-2, 2, 0).$$ [4]

(c) A pair of cubes are glued together so that a face of one is flush with a face of the other.



Each of the ten unglued faces is to be coloured orange or green. Determine how many rotational equivalence classes of such colourings there are [5]

Part IIB

Do not attempt more than two questions from Part IIB.

Question 15

Let G be a finite group and let $Z(G)$ be the centre of G.

(a) Prove that $Z(G)$ is a normal subgroup of G. [7]

(b) Prove Theorem 4.3, page 37 of the Handbook: that if the quotient group $G/Z(G)$ is cyclic, then G is Abelian. [5]

(c) Give an example of a non-Abelian group G with a proper normal subgroup N such that the quotient group G/N is cyclic. [3]

Question 16

Let G be a finite group, let H be a subgroup of G with index k (where $k \geq 2$) and let X be the set of left cosets of H in G.

(a) Prove that $g \wedge xH = gxH$ defines an action of G on the set X. [4]

(b) Explain how the result of part (a) produces a homomorphism o from G to the permutation group S_k. [4]

(c) Prove that the kernel of o is a normal subgroup of G contained in H. [4]

(d) Suppose that, in addition, G is a simple group, i.e. G has no proper non-trivial normal subgroups. Prove that the order of G divides $k!$. [3]

Question 17

Let G be a group of order 45.

(a) For each prime p dividing 45, find the number of Sylow p-subgroups of G, justifying your answers. [6]

(b) For each such G, express G as a non-trivial direct product, giving reasons for your answer, and hence show that any group of order 45 must be Abelian. [9]

[END OF QUESTION PAPER]

Please complete the identification grid below.

Examination No.								
Surname (BLOCK LETTERS)					Initials			
Personal Identifier								

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Figure Sheet

Answer Sheet for Questions 7 and 13

Question 7

Please read carefully the instructions given for Question 7 of the question paper

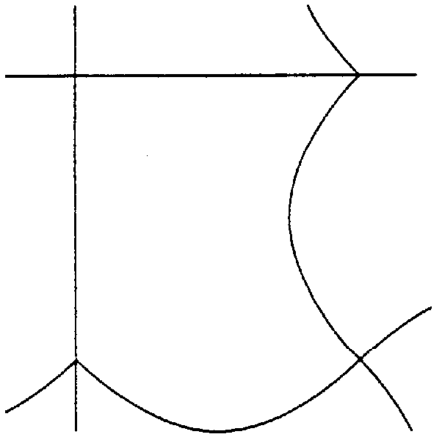

Figure 1

For examiner's use	

Turn over

Question 13
Please read carefully the instructions given for Question 13 of the question paper

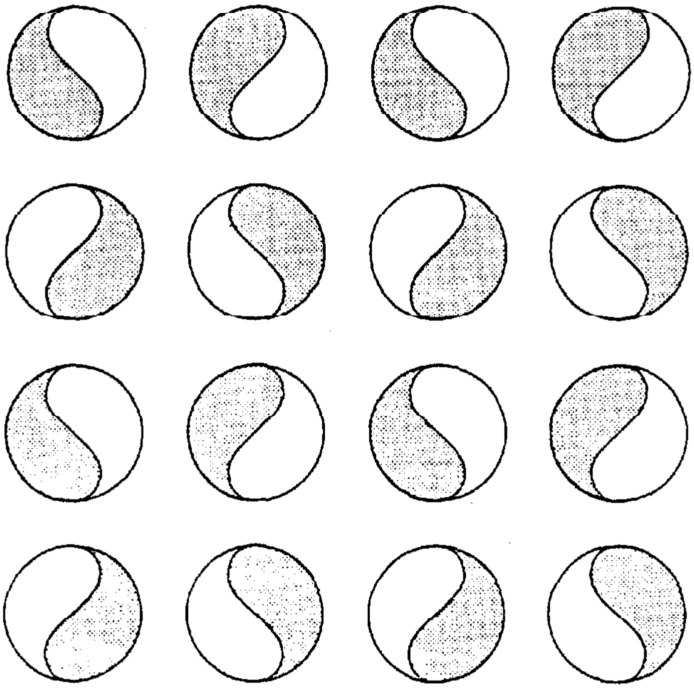

Figure 2

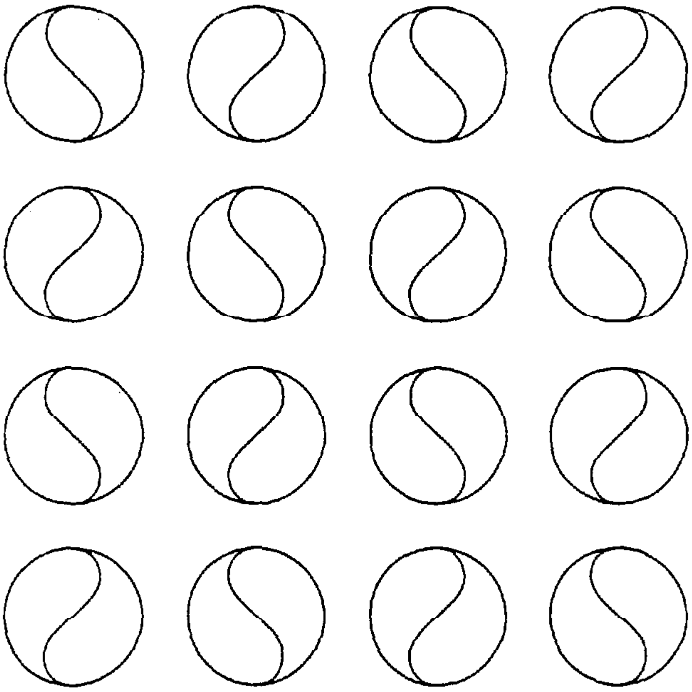

Figure 3

For examiner's use	