M336/W

Third Level Course Examination 1995

Groups and Geometry

Friday, 27th October, 1995 10.00am–1.00pm

Time allowed: 3 hours

There are TWO parts to this examination. You should attempt **BOTH**.

You may attempt **ALL** the questions in Part I. You should answer not more than **THREE** questions from Part II, including not more than **TWO** questions from Part IIA and not more than **TWO** questions from Part IIB.

You are advised to spend about 90 minutes on Part I and 80 minutes on Part II, and to leave yourself about 10 minutes for checking. Part I carries 55% of the total marks while Part II carries 45% of the total marks.

You are advised to show all your working and to give reasons for all your answers unless a question is explicitly phrased 'Write down...'. You should begin each answer on a new page of the answer book.

Note: a Figure Sheet is provided with this question paper for use in answering Questions 7, 11 and 13.

At the end of the examination

Check that you have written your name, examination number and personal identifier on each answer book used and on any separate sheets used. **Failure to do so may mean that your paper cannot be identified.** Attach all your answer books and the Figure Sheet together, using the fastener provided.

Write the numbers of the questions that you have attempted in the spaces provided on the front of the answer book.

Part I

You may attempt **ALL** *the questions in this part and are advised to spend about* **90 minutes** *on it.*

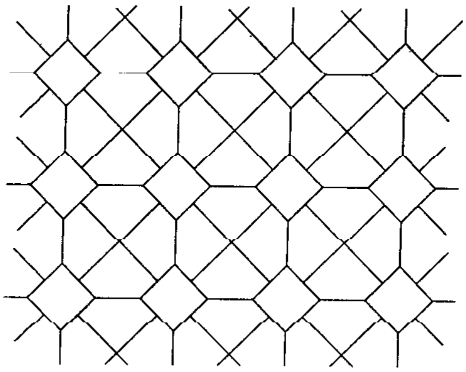
The marks for each question are given beside the question.

Question 1

(a) Find the inverse of the affine transformation $f = t[\mathbf{p}]\lambda[\mathbf{A}]$, where

$$\mathbf{p} = (1, 2) \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ [3]

(b) Write down all the tile types and vertex types in the following tiling.



[2]

Question 2

The dihedral group D_6 is given by the presentation

$$D_6 = \langle r, s : r^6 = e, s^2 = e, sr = r^5 s (= r^{-1}s) \rangle,$$

and the elements are written in standard form as

$$r^m \quad \text{and} \quad r^m s, \quad m = 0, 1, \ldots, 5.$$

(a) Express the product

$$(r^2 s)(r^4 s)$$

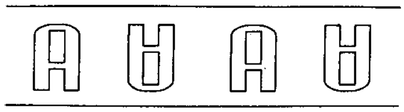
in standard form. [2]

(b) Show that none of the elements of the form $r^m s$ commutes with r. [2]

(c) Write down a non-identity element of D_6 which commutes with *every* element of D_6. (You need not give any justification for your answer.) [1]

Question 3

(a) Use the algorithm of *Unit IB3* to determine the type of the following frieze $\mathcal{F}$.



[4]

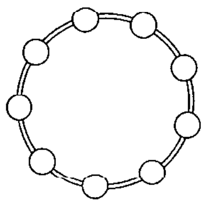
(b) Write down the type of the frieze that would result from $\mathcal{F}$ if every alternate motif (i.e. every inverted 'A') were coloured red, the non-inverted 'A's remaining black. (You need not give a justification for your answer.) [1]

Question 4

Let G be a group and let H be a subgroup of G, of index 2 in G. Prove that H is a normal subgroup of G. [5]

Question 5

Nine glass beads are to be threaded onto a key ring, as shown in the figure below.



Red and green beads are available. How many essentially different possibilities are there? (Two such possibilities are essentially the same if, after they have been in somebody's bag or pocket for some time, they cannot be told apart.)

You should find an arithmetic expression for the number, then evaluate it. [5]

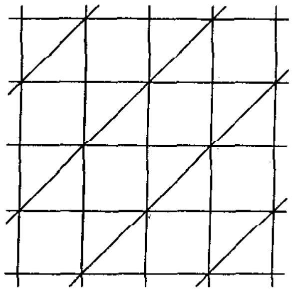
Question 6

Use the Euclidean Algorithm to find hcf$\{234, 456\}$.

[N.B. You *must* use the algorithm; *no marks* will be given for merely factorizing the two numbers.] [5]

Question 7

Figure 1 on the Figure Sheet is an enlarged copy of the figure below, which depicts part of a tiling $\mathcal{T}$.



(a) Write down $n_t(\mathcal{T})$ and $n_v(\mathcal{T})$. [1]

(b) On Figure 1 of the Figure Sheet, indicate:

(i) the translational tile orbits by placing circled numbers in the tiles;

(ii) the translational vertex orbits by placing uncircled numbers by the vertices. [2]

(c) Using your numbering from part (b), draw the tile-vertex diagram for $\mathcal{T}$. [2]

Question 8

The finitely presented Abelian group A is defined by

$$A = \langle a, b, c : 2a + 4b + 6c = 0, 4a + 10b + 12c = 0 \rangle.$$

(a) Write down the integer matrix representing A. [1]

(b) Use the Reduction Algorithm to reduce the matrix to diagonal form. [3]

(c) Hence express A as a direct product of non-trivial cyclic groups in canonical form. [1]

Question 9

Let $\mathbf{a} = (1, 0)$ and $\mathbf{b} = (1/2, \sqrt{3}/2)$.

(a) Write down the lattice type of $L(\mathbf{a}, \mathbf{b})$. (You need not give any justification.) [1]

(b) Show that $L(\mathbf{a}, 3\mathbf{b}) = L(3\mathbf{b} - 2\mathbf{a}, 3\mathbf{b} - \mathbf{a})$, and hence determine the lattice type of $L(\mathbf{a}, 3\mathbf{b})$. [3]

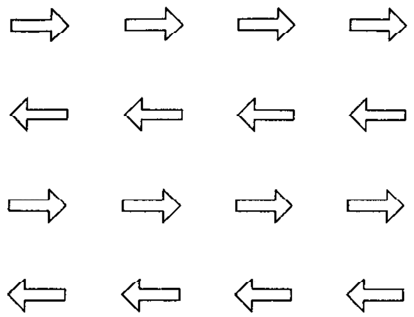
(c) Write down the lattice type of $L(\mathbf{a}, 2\mathbf{b})$. (You need not give any justification.) [1]

Question 10

Let G be a non-trivial p-group, where p is prime. By using the Orbit-Stabilizer Theorem and the class equation of G, prove that G has a non-trivial centre. [5]

Question 11

Consider the wallpaper pattern W below, a copy of which is reproduced as Figure 2 on the Figure Sheet.



(a) On Figure 2 on the Figure Sheet:

(i) draw a basic rectangle of W;

(ii) within your basic rectangle, shade in a generating region. [3]

(b) Write down the wallpaper type of W. (You need not give a justification.) [2]

Part II

You may attempt not more than **THREE** *questions from this part, and you are advised to spend about* **80 minutes** *on it.*

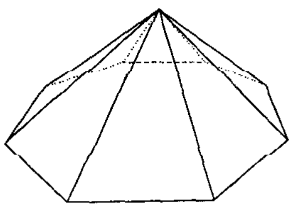
You may choose not more than **TWO** *of your three questions from Part IIA and not more than* **TWO** *from Part IIB.*

Each question carries 15% of the marks for the whole examination, and an indication of the allocation of marks within each question is given beside the question.

Part IIA
Do not attempt more than two questions from Part IIA.

Question 12

In this question we consider a pyramid on a rectangular octagonal base. (This object is the octagonal analogue of the hexagonal pyramid in the video programme VC2A.)



The group of rotations of the pyramid is

$$G = \{r^n : n = 0, 1, \ldots, 7\},$$

where $r = r[\pi/4]$. The triangular faces are to be coloured purple (P) or yellow (Y). The underside is uncoloured.

(a) Find the cycle index of G considered as acting on the triangular faces. [5]

(b) Write down the pattern inventory, and determine the numbers of equivalence classes of colourings with:

(i) no yellow faces;

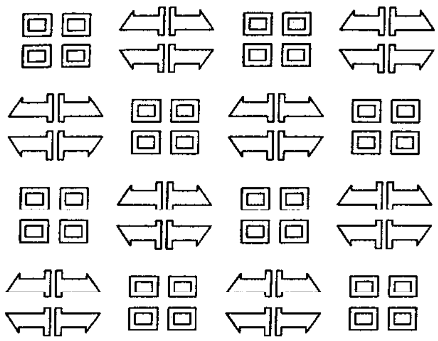
(ii) one yellow face;

(iii) two yellow faces. [6]

(c) Find the total number of equivalence classes of colourings. [4]

Question 13

Consider the following wallpaper pattern W, two enlarged copies of which are reproduced as Figures 3 and 4 on the Figure Sheet.



(a) Carry out the algorithm described in the audio tape for *Unit GE4* to determine the type of W. You should explain carefully, at each stage, your answer to the question posed by the algorithm. [5]

(b) Using Figure 3 of the Figure Sheet, indicate with a clearly visible cross or dot *one* representative of *each* orbit of non-trivial rotation centres under the group $\Gamma(W)$. [3]

(c) Draw a basic parallelogram on Figure 3 of the Figure Sheet, and shade in a generating region. [3]

(d) By shading some of the motifs and leaving others unshaded, it is possible to alter the wallpaper type of the pattern. EITHER turn Figure 4 on the Figure Sheet into a pattern of type $p2mg$ by such a shading process, OR describe *very clearly* which features should be shaded to achieve this result. [4]

Question 14

Find the type of each of the following Bravais lattices.

(a) $L(\mathbf{a}, \mathbf{b}, \mathbf{c})$, where

$$\mathbf{a} = (1, 1, \sqrt{2}), \quad \mathbf{b} = (-1, -1, \sqrt{2}), \quad \mathbf{c} = (\sqrt{2}, -\sqrt{2}, 0).$$ [3]

(b) $L(\mathbf{a}, \mathbf{b}, \mathbf{c})$, where

$$\mathbf{a} = (1, 3, 0), \quad \mathbf{b} = (1, -3, 0), \quad \mathbf{c} = (2, 0, 2).$$ [3]

(c) $L(\mathbf{a}, \mathbf{b}, \mathbf{c})$, where

$$\mathbf{a} = (1, 0, 0), \quad \mathbf{b} = (0, 1, 0), \quad \mathbf{c} = (\sqrt{5}, \sqrt{6}, \sqrt{7}).$$ [3]

(d) $L(\mathbf{a}, \mathbf{b}, \mathbf{c})$, where

$$\mathbf{a} = (6, 8, 0), \quad \mathbf{b} = (8, -6, 0), \quad \mathbf{c} = (7, 1, 5).$$ [3]

(e) $L(\mathbf{a}, \mathbf{b}, \mathbf{c})$, where

$$\mathbf{a} = (1, \sqrt{3}, 0), \quad \mathbf{b} = (-1, \sqrt{3}, 0), \quad \mathbf{c} = (0, 2\sqrt{3}, 1).$$ [3]

Part IIB

Do not attempt more than two questions from Part IIB.

Question 15

(a) Let G be a group, and let H and K be normal subgroups of G. Show that the subgroup $H \cap K$ of G is normal. [6]

(b) Give an example to show that it is possible for H and K to be *non-normal* subgroups of G, but for $H \cap K$ nevertheless to be a *normal* subgroup of G. [5]

(c) If H and K are subgroups of a group G, is it possible for $H \cap K$ to be a normal subgroup both of H and of K but *not* of G? If so, give an example; if not, give a reason. [4]

Question 16

(a) Find all Abelian groups of order 72 (up to isomorphism), expressing each of them *both* in canonical *and* in p-primary form. [9]

(b) Determine which of the groups found in part (a) have the property that *every* subgroup of order 4 is cyclic. [6]

Question 17

Let G be a group of order 440.

(a) For each prime p dividing 440, find the possible numbers of Sylow p-subgroups of G as permitted by the Sylow Theorems. [5]

(b) Use results from *Units GR4* and *GR5* to show that G has a normal subgroup of order 11 and (not necessarily normal) subgroups of orders 22, 44, 55 and 88. [6]

(c) By considering the conjugacy action of G on the set of Sylow 5-subgroups of G, show that *either* G has a normal subgroup of order 5 *or* G has a (not necessarily normal) subgroup of order 40. [4]

[END OF QUESTION PAPER]