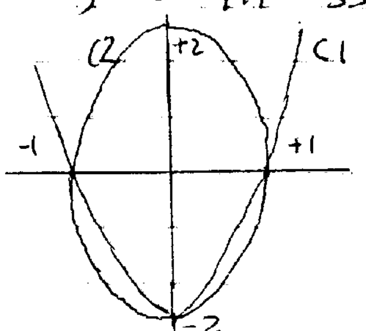
MS221 1997 Exam Answers

1. $1000x = 345.345\,345\,345 = 345 + x$, $999x = 345$, $x = \frac{345}{999} = \frac{115}{333}$



2. C1 $y = 2x^2 - 2$ parabola

C2 $\frac{x^2}{1} + \frac{y^2}{4} = 1$ ellipse

For C1, $y = 0$, $x = \pm 1$, $x = 0$, $y = -2$

For C2, $y = 0$, $x = \pm 1$, $x = 0$, $y = \pm 2$

3(a)(i) trans_{10} (ii) trans_0 (do nothing, or identity transformation)

(b)(i) $\text{ref}_a \text{ref}_b = \text{trans}_{2a-2b}$, for $\text{ref}_p \text{ref}_4 = \text{trans}_6$, $6 = 2p - 2\times 4$, $p = 7$

(ii) $\text{ref}_4 \text{ref}_p = \text{trans}_6$, $6 = 2\times 4 - 2p$, $p = 1$

4. $\frac{2}{3}\begin{pmatrix}1\\3\\-5\end{pmatrix} + \frac{1}{3}\begin{pmatrix}4\\-3\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}6\\3\\-9\end{pmatrix} = \begin{pmatrix}2\\1\\-3\end{pmatrix}$

5. Choose $-1 \le x_1 \le x_2 \le 1$, $f(x_2) - f(x_1) = x_2^2 - 3x_2 - (x_1^2 - 3x_1)$

$= (x_2^2 - x_1^2) - 3(x_2 - x_1) = (x_2 + x_1)(x_2 - x_1) - 3(x_2 - x_1)$

$= (x_2 - x_1)(x_2 + x_1 - 3)$. Now, $(x_2 - x_1) > 0$, but $(x_2 + x_1 - 3) < 0$, since max^m $x_2 + x_1$ is 2, hence $f(x_2) - f(x_1) < 0$, and the function f is not increasing on $[-1, 1]$

6(a) 4×5 grid gives a total number of routes $A \to B$ of ${}^9C_4 = 126$

(b) Routes $A \to X = {}^3C_1 = 3$, routes $X \to B = {}^6C_3 = 20$

Routes $A \to B$ via $X = 3\times 20 = 60$. Proportion of total $= 60/126 = 10/21$

7. Fix points are solutions of $x^2 - 4x + 6 = x$, i.e. $x^2 - 5x + 6 = 0$, which gives $x = 2, 3$

8(a) Eigenvector matrix $= Q = \begin{pmatrix}2 & 1\\-1 & -1\end{pmatrix}$, determinant $= -1$, so $Q^{-1} = \begin{pmatrix}1 & 1\\-1 & -2\end{pmatrix}$

hence $M = \begin{pmatrix}2 & 1\\-1 & -1\end{pmatrix}\begin{pmatrix}2 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\-1 & -2\end{pmatrix}$ where D is formed from the eigenvalues

(b) $M^{10} = \begin{pmatrix}2 & 1\\-1 & -1\end{pmatrix}\begin{pmatrix}2^{10} & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\-1 & -2\end{pmatrix}$

$= \begin{pmatrix}2 & 1\\-1 & -1\end{pmatrix}\begin{pmatrix}1024 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\-1 & -2\end{pmatrix}$

9(a) $f'(x) = \frac{1}{x}\cos(\ln(2x))$ composite rule

(b) $\frac{1}{2}\left(\frac{x}{\sin x}\right)^{-1/2} \cdot \left(\frac{\sin x - x\cos x}{\sin^2 x}\right)$ composite and quotient rules.

10(a) $\int \frac{1+x}{\sqrt{x}}\,dx = \int (x^{-1/2} + x^{1/2})\,dx = 2x^{1/2} + \frac{2}{3}x^{3/2} + C = 2\sqrt{x} + \frac{2}{3}x\sqrt{x} + C$

(b) $du = \cos x\,dx$, Integral $= \int \frac{du}{u} = \ln u + C = \ln(3+\sin x) + C$.

11. $f(x) = \frac{1}{2}(1 - x + x^2 - x^3 + x^4 - x^5 + x^6 \ldots + 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 \ldots$
$= \frac{1}{2}(2 + 2x^2 + 2x^4 + 2x^6 \ldots) = 1 + x^2 + x^4 + x^6 \ldots$

12(a) Separate variables: $\frac{dy}{y} = \cos x\,dx$, $\ln y = \sin x + A$, $y = Ce^{\sin x}$.

(b) $x = 0 \to \sin x = 0$, $e^{\sin x} = 1$, so $C = 1$. $y = e^{\sin x}$

13.(a) $z = \sqrt{3} + i = a + bi$, $r = \sqrt{a^2+b^2} = 2$, $\theta = \cos^{-1}\frac{a}{r} = \cos^{-1}\frac{\sqrt{3}}{2} = \pi/6$, $\langle 2, \pi/6 \rangle$

(b) $w = (\langle 2, \pi/6 \rangle)^{1/3} = \langle \sqrt[3]{2}, \frac{1}{3}(2m\pi) + \frac{\pi}{18} \rangle$ with $m = 0, 1, 2$. Roots are $\langle \sqrt[3]{2}, \pi/18 \rangle$, $\langle \sqrt[3]{2}, \frac{13\pi}{18} \rangle$, $\langle \sqrt[3]{2}, \frac{25\pi}{18} \rangle$

14.(a) 11 (b) 2, 4, 6, 8, 10 (all the even numbers)

15.(a) r (b) yes, because the table is symmetrical

(c) r and s. $(G, *)$ is order 4, Abelian, 2 self-inverses, hence is isomorphic to $(\mathbb{Z}_4, +_4)$

16(a) (B) is false

(b) take $p = 2$, $q = 3$, $pq = 6$. We have pq is even, but q is odd.

(c) This is proof by counter-example

17(a) We construct the next larger or smaller pentagon by an identical method each time.

(b) Original has side 1, diagonal ϕ. For new pentagon, diagonal is 1, plus 2 extensions – one each side – of ϕ. So, diagonal is $1 + 2\phi = \phi + \phi^2$, since $\phi^2 - \phi - 1 = 0$. So, scale factor $= \frac{\phi + \phi^2}{\phi} = 1 + \phi = \phi^2$, again since $\phi^2 - \phi - 1 = 0$. Scale factor is ϕ^2.

(c)(i) The larger root in $x^2-x-1=0$. Used in (b) above

(ii) Ratio of sides of golden rectangle. We can produce larger or smaller golden rectangles from an original using identical methods, as for pentagons

(d) See unit - but essentially non of these examples are exact, & rely on interpretation.

18 (a)

Power	matrix
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

$(1+x)^5 = 1+5x+10x^2+10x^3+5x^4+1.x^5$

Coefficients correspond to bottom matrix.

(b)(i) Each coefficient in $(1+2x)^{k+1}$ is the sum of the coefficient of the same power in M, and twice the sum of the next lower power. Need only change the second line in F to:-

Form a matrix C by inserting 0 to the left and doubling. Remainder of the algorithm is fine.

(b)(ii)

Power	matrix
0	1
1	1 2
2	1 4 4
3	1 6 12 8
4	1 8 24 32 16

$(1+2x)^4 = 1+8x+24x^2+32x^3+16x^4$. Coefficients agree with M

19 (a) $A(x+h)-A(x) = \delta A \doteqdot$ width $\times$ ht of rectangle $\doteqdot \delta x \times \sin\theta$, hence $\frac{\delta A}{\delta x} \doteqdot \sin\theta$.

In limit, as $\delta x \to 0$, $\frac{dA}{dx} = \sin\theta$

(b) $x = -\cos\theta$. $\frac{dA}{d\theta} = \frac{dA}{dx}\cdot\frac{dx}{d\theta} = \sin\theta.\sin\theta = \sin^2\theta$. When $\theta=0$, A becomes the point $(-1,0)$ which has zero area.

(c) $A = \int \sin^2\theta\, d\theta = \int \frac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$, $A=0$ when $\theta=0$, so $C=0$

hence $A = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta$. (ii) For semicircle $\theta=\pi$, hence $A = \pi/2$.

(d)(i) $2/3$ of semicircle area is $\frac{2}{3}\cdot\frac{\pi}{2} = \pi/3$, ie solve $\frac{1}{3}\pi = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta$, $\theta - \frac{1}{2}\sin 2\theta - \frac{2}{3}\pi = 0$

(ii) N-R formula is $\theta_{n+1} = \theta_n - \frac{f(\theta_n)}{f'(\theta_n)}$, where $f(\theta_n) = \theta - \frac{1}{2}\sin 2\theta - \frac{2}{3}\pi$, and $f'(\theta_n) = 1-\cos 2\theta$

(e)(i) 1·8329 - 2 successive iterations round to this value. (ii) $x = -\cos 1{\cdot}8329 = 0{\cdot}265$ rads.

20 (a) $4 = 26 - 2\times 11$

$3 = 11 - 2\times 4$

$1 = 4 - 1\times 3 = 4 - (11 - 2\times 4)$

$= -11 + 3\times 4$

$= -11 + 3(26 - 2\times 11)$

$= -7\times 11$ (all mod 26)

$-7 \equiv 19 \pmod{26}$

so required inverse is 19

(b)

ciphers in message	$\times 19$	multiple of 26	Value in $\mathbb{Z}_{26}$	Letter
10	190	182	8	H
11	209	208	1	A
12	228	208	20	T
15	285	260	25	Y
16	304	286	18	R
18	342	338	4	D
20	380	364	16	P
21	399	390	9	I
22	418	416	2	B

Message is HAPPY BIRTHDAY