Question 7

(i) [*5 marks*]

$$\alpha^a_{bc}\theta^c \wedge \theta^b = 0$$

and therefore

$$\sum_{b<c} (\alpha^a_{bc} - \alpha^a_{cb})\, \theta^b \wedge \theta^c = 0,$$

whence $\alpha^a_{bc} = \alpha^a_{cb}$, since $\{\theta^b \wedge \theta^c \mid b < c\}$ is a basis for the 2-forms. Further

$$\begin{aligned}
\alpha^d_{ac}\delta_{db} + \alpha^d_{bc}\delta_{ad} &= 0 \\
\alpha^d_{ba}\delta_{dc} + \alpha^d_{ca}\delta_{bd} &= 0 \\
\alpha^d_{cb}\delta_{da} + \alpha^d_{ab}\delta_{cd} &= 0.
\end{aligned}$$

Add the first two, subtract the third, and use the symmetry of δ_{ef}: one obtains

$$(\alpha^d_{ac} + \alpha^d_{ca})\delta_{bd} + (\alpha^d_{bc} - \alpha^d_{cb})\delta_{ad} + (\alpha^d_{ba} - \alpha^d_{ab})\delta_{cd} = 0,$$

whence $2\alpha^d_{ac}\delta_{bd} = 0$, which implies that $\alpha^d_{ac} = 0$, which in turn implies that $\alpha^a_b = 0$.

(ii) [*2 marks*]

If ω^a_b and $\hat\omega^a_b$ both satisfy the given conditions then

$$\begin{aligned}
&(\omega^a_b - \hat\omega^a_b) \wedge \theta^b = 0 \\
&(\omega^d_a - \hat\omega^d_a)\delta_{db} + (\omega^d_b - \hat\omega^d_b)\delta_{ad} = 0
\end{aligned}$$

and so, from part (i), $\omega^a_b - \hat\omega^a_b = 0$, and the ω^a_b are uniquely determined.

(iii) [*5 marks*]

$$\langle U_a, \theta^b \rangle = \delta^b_a.$$

Take the Lie derivative of this equation with respect to X:

$$\langle \mathcal{L}_X U_a, \theta^b \rangle + \langle U_a, \mathcal{L}_X \theta^b \rangle = 0,$$

whence

$$\langle [X, U_a], \theta^b \rangle = -\langle U_a, \mathcal{L}_X \theta^b \rangle = -\xi^b_a$$

as required. If X is a Killing field, since $\{U_a\}$ is an orthonormal basis

$$\begin{aligned}
0 = X(g(U_a, U_b)) &= g([X, U_a], U_b) + g(U_a, [X, U_b]) \\
&= -\, g(\xi^d_a U_d, U_b) - g(U_a, \xi^d_b U_d) \\
&= -\, \xi^d_a \delta_{db} - \xi^d_b \delta_{ad}
\end{aligned}$$

as required.

(iv) [*8 marks*]

Taking the Lie derivative of the first structure equations gives

$$\begin{aligned}
d(\mathcal{L}_X\theta^a) + (\mathcal{L}_X\omega^a_b) \wedge \theta^b + \omega^a_b \wedge (\mathcal{L}_X\theta^b) &= 0 \\
= d(\xi^a_b\theta^b) + (\mathcal{L}_X\omega^a_b) \wedge \theta^b + \omega^a_c \wedge \xi^c_b\theta^b& \\
= d\xi^a_b\theta^b - \xi^a_c\omega^c_b \wedge \theta^b + (\mathcal{L}_X\omega^a_b) \wedge \theta^b + (\omega^a_c\xi^c_b) \wedge \theta^b& \\
= (\mathcal{L}_X\omega^a_b + d\xi^a_b + \omega^a_c\xi^c_b - \xi^a_c\omega^c_b) \wedge \theta^b.&
\end{aligned}$$