The connection 1-forms are

$$\omega^1_2 = e^{x^1}\,dx^2 \qquad \omega^2_1 = -e^{x^1}\,dx^2 \qquad \omega^1_1 = \omega^2_2 = 0.$$

By evaluating $d\theta^a + \omega^a_b \wedge \theta^b$ for the dual basis of 1-forms $\{\theta^a\}$, given by

$$\theta^1 = dx^1 + e^{x^1}\,dx^2 \qquad \theta^2 = dx^1 - e^{x^1}\,dx^2,$$

show that the connection has zero torsion. Compute $\nabla_{\partial_1}\partial_2$ and $\nabla_{\partial_2}\partial_1$.

Question 2

Let $\mathcal{V}$ be a vector space, $\mathcal{V}^*$ its dual.

(i) Explain what its means to say that a map $T : \mathcal{V}^* \times \mathcal{V} \to \mathbf{R}$ is multilinear (or, in this case, bilinear).

(ii) For any $v \in \mathcal{V}$ and $\alpha \in \mathcal{V}^*$ let $v \otimes \alpha$ be the map $\mathcal{V}^* \times \mathcal{V} \to \mathbf{R}$ defined by

$$v \otimes \alpha(\beta, w) = \langle v, \beta\rangle\langle w, \alpha\rangle \qquad \beta \in \mathcal{V}^*, \qquad w \in \mathcal{V}.$$

Show that $v \otimes \alpha$ is bilinear.

(iii) Explain how the set $\mathcal{T}$ of billinear maps $\mathcal{V}^* \times \mathcal{V} \to \mathbf{R}$ may be given the structure of a vector space. (You are *not* required to verify in detail that the vector space axioms are satisfied.) Show that if $\{e_a\}$ is a basis for $\mathcal{V}$ and $\{\theta^a\}$ is the dual basis for $\mathcal{V}^*$ then $\{e_a \otimes \theta^b\}$ is a basis for $\mathcal{T}$, and hence give the dimension of $\mathcal{T}$.

(iv) For any linear map $\lambda : \mathcal{V} \to \mathcal{V}$ define $\tilde{\lambda} : \mathcal{V}^* \times \mathcal{V} \to \mathbf{R}$ by

$$\tilde{\lambda}(\beta, w) = \langle \lambda(w), \beta\rangle.$$

Show that the map $\lambda \to \tilde{\lambda}$ is an isomorphism of the space of linear maps $\mathcal{V} \to \mathcal{V}$ with $\mathcal{T}$.

Question 3

(i) Let $\mathcal{M}$ be a topological manifold. Explain what is meant by a chart on $\mathcal{M}$; by saying that two charts on $\mathcal{M}$ are smoothly related; and by a smooth atlas for $\mathcal{M}$.

(ii) A differentiable manifold $\mathcal{N}$ having been given in advance, let $\mathcal{M}$ be the set of all tangent vectors at all points of $\mathcal{N}$, that is, the union of all $T_x\mathcal{N}$ for all $x \in \mathcal{N}$. For any open set $\mathcal{O}$ in $\mathcal{N}$ let $\tilde{\mathcal{O}}$ be the subset of $\mathcal{M}$ consisting of all tangent vectors to $\mathcal{N}$ at points of $\mathcal{O}$, that is, the union of all $T_x\mathcal{N}$ for all $x \in \mathcal{O}$. Assume that $\mathcal{M}$ may be given the structure of a topological manifold in such a way that $\tilde{\mathcal{O}}$ is an open set. For any chart $(\mathcal{P}, \psi)$ on $\mathcal{N}$ define a map $\tilde{\psi}$ on $\tilde{\mathcal{P}}$ by

$$\tilde{\psi}(u) = (x^a, u^a)$$

for any tangent vector $u \in \tilde{\mathcal{P}}$, where (x^a) are the coordinates, as determined by ψ, of the point $x \in \mathcal{P}$ where u is defined, and u^a are the components of u with respect to the coordinate vectors at x. Assume that $(\tilde{\mathcal{P}}, \tilde{\psi})$ is a chart for $\mathcal{M}$.

By considering the effect of a coordinate transformation in $\mathcal{N}$ on the components of tangent vectors, show that if $\{(\mathcal{P}_\alpha, \psi_\alpha)\}$ is a smooth atlas for $\mathcal{N}$ then $\{(\tilde{\mathcal{P}}_\alpha, \tilde{\psi}_\alpha)\}$ is a smooth atlas for $\mathcal{M}$. Conclude that $\mathcal{M}$ is a differentiable manifold, and give its dimension in terms of the dimension of $\mathcal{N}$.

(iii) Let $\pi : \mathcal{M} \to \mathcal{N}$ be the map that assigns to each tangent vector the point of $\mathcal{N}$ at which it is defined, so that $u \in T_{\pi(u)}\mathcal{N}$. Show that π is a smooth surjective submersion.