Question 6

Let ∇ be a connection on a differentiable manifold $\mathcal{M}$.

(i) Explain what it means to say that ∇ is symmetric. Define the connection coefficients of ∇ with respect to a coordinate system on $\mathcal{M}$, and give the necessary and sufficient conditions, in terms of the connection coefficients, for ∇ to be symmetric.

(ii) Define the curvature R of the connection. Show that, for any smooth function f and vector fields U, V and W on $\mathcal{M}$,

$$R(U,V)fW = fR(U,V)W.$$

What other properties of R is it necessary to check to confirm that it is a tensor field? (You are *not* asked to check these further properties.)

(iii) Write down an expression for the components of R, with respect to a coordinate system, in terms of the connection coefficients. (You may assume that R is a tensor field.)

(iv) State and prove the first Bianchi identity for the curvature under the assumption that the connection is symmetric.

Question 7

(i) Let $\{\theta^a\}$ be a local basis of 1-forms on a manifold $\mathcal{M}$ of dimension m. Suppose that there are m^2 1-forms α^a_b such that

$$\alpha^a_b \wedge \theta^b = 0.$$

Set $\alpha^a_b = \alpha^a_{bc}\theta^c$. Show that

$$\alpha^a_{cb} = \alpha^a_{bc}.$$

Suppose further that

$$\alpha^d_a \delta db + \alpha^d_b \delta ad = 0.$$

By considering the equation $\alpha^d_{ac}\delta db + \alpha^d_{bc}\delta ad = 0$ and the two further equations obtained by cyclically permuting the suffices a, b and c, show that when both conditions on the α^a_b hold

$$\alpha^a_b = 0.$$

(ii) Let $\{U_a\}$ be a local orthonormal basis of vector fields on a Riemannian manifold $\mathcal{M}$, with dual basis of 1-forms $\{\theta^a\}$. The connection 1-forms ω^a_b, associated with these bases, for the Levi-Civita connection on $\mathcal{M}$ satisfy the first structure equations

$$d\theta^a + \omega^a_b \wedge \theta^b = 0$$

and the skew-symmetry condition

$$\omega^d_a \delta db + \omega^d_b \delta ad = 0.$$

Deduce from the result of part (i) that the ω^a_b are uniquely determined by these conditions.

(iii) Suppose that X is a Killing field on $\mathcal{M}$. Define functions ξ^a_b by

$$\mathcal{L}_X \theta^a = \xi^a_b \theta^b.$$

Show that

$$[X, U_a] = -\xi^b_a U_b$$

and deduce that the ξ^a_b satisfy

$$\xi^d_a \delta db + \xi^d_b \delta ad = 0.$$

(iv) By taking the Lie derivative of the first structure equations with respect to X and using the result of part (i) show that

$$\mathcal{L}_X \omega^a_b + d\xi^a_b + \omega^a_c \xi^c_b - \xi^a_c \omega^c_b = 0.$$