Question 2

(i) [*2 marks*]

For T to be bilinear it must be linear in each argument separately: that is, for every $\alpha, \alpha_1, \alpha_2 \in \mathcal{V}^*$, $v, v_1, v_2 \in \mathcal{V}$, $k_1, k_2 \in \mathbf{R}$,

$$T(k_1\alpha_1 + k_2\alpha_2, v) = k_1T(\alpha_1, v) + k_2T(\alpha_2, v)$$
$$T(\alpha, k_1v_1 + k_2v_2) = k_1T(\alpha, v_1) + k_2T(\alpha, v_2).$$

(ii) [*4 marks*]

$$\begin{aligned} v\otimes\alpha(k_1\beta_1 + k_2\beta_2, w) &= \langle v, k_1\beta_1 + k_2\beta_2\rangle\langle w, \alpha\rangle \\ &= k_1\langle v, \beta_1\rangle\langle w, \alpha\rangle + k_2\langle v, \beta_2\rangle\langle w, \alpha\rangle \\ &= k_1 v\otimes\alpha(\beta_1, w) + k_2 v\otimes\alpha(\beta_2, w) \end{aligned}$$

and

$$\begin{aligned} v\otimes\alpha(\beta, k_1w_1 + k_2w_2) &= \langle v, \beta\rangle\langle k_1w_1 + k_2w_2, \alpha\rangle \\ &= k_1\langle v, \beta\rangle\langle w_1, \alpha\rangle + k_2\langle v, \beta\rangle\langle w_2, \alpha\rangle \\ &= k_1 v\otimes\alpha(\beta, w_1) + k_2 v\otimes\alpha(\beta, w_2); \end{aligned}$$

so $v\otimes\alpha$ is bilinear.

(iii) [*8 marks*]

The main requirement is to define addition of two bilinear maps and multiplication of a bilinear map by a scalar: this is done pointwise, as follows:

$$(T_1 + T_2)(\beta, w) = T_1(\beta, w) + T_2(\beta, w)$$
$$(kT)(\beta, w) = k(T(\beta, w)).$$

With these definitions $T_1 + T_2$ and kT are bilinear, and $\mathcal{T}$ becomes a vector space, the axioms being satisfied essentially because they are satisfied in $\mathbf{R}$. The bilinear map $e_a\otimes\theta^b$ satisfies

$$e_a\otimes\theta^b(\theta^c, e_d) = \delta_a^c\delta_d^b.$$

It follows that the $e_a\otimes\theta^b$ are linearly independent. For suppose that

$$k_b^a e_a\otimes\theta^b = 0$$

for some scalars k_b^a (summation convention in force). Then evaluating on (θ^c, e_d) gives

$$k_b^a e_a\otimes\theta^b(\theta^c, e_d) = k_b^a\delta_a^c\delta_d^b = k_d^c = 0.$$

This holds for all c, d, so the coefficients are all zero, and the $e_a\otimes\theta^b$ are therefore linearly independent. Moreover, they span $\mathcal{T}$. For if $T\in\mathcal{T}$, set $T(\theta^a, e_b) = T_b^a$ and consider the bilinear map $\hat{T}$ defined by

$$\hat{T} = T_b^a e_a\otimes\theta^b.$$

For any $\beta\in\mathcal{V}^*$, $w\in\mathcal{V}$

$$\hat{T}(\beta, w) = T_b^a\langle e_a, \beta\rangle\langle w, \theta^b\rangle = T_b^a\beta_a w^b$$

where $\beta = \beta_a\theta^a$, $w = w^be_b$. But

$$T(\beta, w) = T(\beta_a\theta^a, w^be_b) = \beta_a w^b T(\theta^a, e_b) = \beta_a w^b T_b^a = \hat{T}(\beta, w).$$