Question 5

(i) [*2 marks*]

$$\mathcal{L}_V\omega = \lim_{t\to 0}\frac{1}{t}\left(\phi_t{}^*\omega - \omega\right);$$

to be explicit

$$(\mathcal{L}_V\omega)_x = \lim_{t\to 0}\left(\phi_t{}^*(\omega_{\phi_t(x)} - \omega_x\right).$$

(ii) [*8 marks*]

$$\begin{aligned}
\mathcal{L}_V(f\alpha) &= \lim_{t\to 0}\frac{1}{t}\left(\phi_t{}^*(f\alpha) - f\alpha\right)\\
&= \lim_{t\to 0}\frac{1}{t}\left((f\circ\phi_t)(\phi_t{}^*\alpha) - f\alpha\right)\\
&= \lim_{t\to 0}\frac{1}{t}(f\circ\phi_t)\left(\phi_t{}^*\alpha - \alpha\right) + \lim_{t\to 0}\frac{1}{t}\left((f\circ\phi_t) - f\right)\alpha\\
&= f\lim_{t\to 0}\frac{1}{t}\left(\phi_t{}^*\alpha - \alpha\right) + \left(\lim_{t\to 0}\frac{1}{t}(f\circ\phi_t - f)\right)\alpha\\
&= f\mathcal{L}_V\alpha + (Vf)\alpha.
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_V(\alpha\wedge\beta) &= \lim_{t\to 0}\frac{1}{t}\left(\phi_t{}^*(\alpha\wedge\beta) - \alpha\wedge\beta\right)\\
&= \lim_{t\to 0}\frac{1}{t}\left((\phi_t{}^*\alpha)\wedge(\phi_t{}^*\beta) - \alpha\wedge\beta\right)\\
&= \lim_{t\to 0}\frac{1}{t}\left((\phi_t{}^*\alpha - \alpha)\wedge(\phi_t{}^*\beta)\right) + \lim_{t\to 0}\frac{1}{t}\left(\alpha\wedge(\phi_t{}^*\beta - \beta)\right)\\
&= \left(\lim_{t\to 0}\frac{1}{t}(\phi_t{}^*\alpha - \alpha)\right)\wedge\beta + \alpha\wedge\left(\lim_{t\to 0}\frac{1}{t}(\phi_t{}^*\beta - \beta)\right)\\
&= (\mathcal{L}_V\alpha)\wedge\beta + \alpha\wedge(\mathcal{L}_V\beta).
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_V(d\alpha) &= \lim_{t\to 0}\frac{1}{t}\left(\phi_t{}^*(d\alpha) - d\alpha\right)\\
&= \lim_{t\to 0}\frac{1}{t}\left(d(\phi_t{}^*\alpha) - d\alpha\right)\\
&= d\left(\lim_{t\to 0}\frac{1}{t}(\phi_t{}^*\alpha - \alpha)\right)\\
&= d\left(\mathcal{L}_V\alpha\right).
\end{aligned}$$

(iii) [*10 marks*]

$$\begin{aligned}
\tilde{\mathcal{L}}_V(f\alpha) &= V\lrcorner d(f\alpha) + d(V\lrcorner f\alpha)\\
&= V\lrcorner(df\wedge\alpha + fd\alpha) + d(f(V\lrcorner\alpha))\\
&= (Vf)\alpha - df\wedge(V\lrcorner\alpha) + f(V\lrcorner d\alpha) + df\wedge(V\lrcorner\alpha) + fd(V\lrcorner\alpha)\\
&= f\tilde{\mathcal{L}}_V\alpha + (Vf)\alpha.
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathcal{L}}_V(\alpha\wedge\beta) &= V\lrcorner d(\alpha\wedge\beta) + d(V\lrcorner(\alpha\wedge\beta))\\
&= V\lrcorner(d\alpha\wedge\beta + (-1)^q\alpha\wedge d\beta)\\
&\quad + d((V\lrcorner\alpha)\wedge\beta + (-1)^q\alpha\wedge(V\lrcorner\beta))\\
&\quad \text{(supposing } \alpha \text{ to be a } q\text{-form)}\\
&= (V\lrcorner d\alpha)\wedge\beta + (-1)^{q+1}d\alpha\wedge(V\lrcorner\beta)\\
&\quad + (-1)^q(V\lrcorner\alpha)\wedge d\beta + (-1)^{2q}\alpha\wedge(V\lrcorner d\beta)
\end{aligned}$$