(c) [4 marks]

$$\begin{aligned}\mathcal{L}_V\alpha &= V(x^1)dx^1 + x^1dV^1 + V(x^2)dx^2 + x^2dV^2\\ &\quad + V(x^3)dx^3 + x^3dV^3\\ &= (x^2 - x^3)dx^1 + x^1(dx^2 - dx^3) + (x^3 - x^1)dx^2 + x^2(dx^3 - dx^1)\\ &\quad + (x^1 - x^2)dx^3 + x^3(dx^1 - dx^2)\\ &= (x^2 - x^3 - x^2 + x^3)dx^1 + (x^1 + x^3 - x^1 - x^3)dx^2\\ &\quad + (-x^1 + x^2 + x^1 - x^2)dx^3\\ &= 0.\end{aligned}$$

(d) [5 marks]

$$\begin{aligned}\alpha \wedge \beta &= x^1x^3x^4dx^1 \wedge dx^2 \wedge dx^3 - x^2x^3x^4dx^1 \wedge dx^2 \wedge dx^4\\ &\quad + x^1x^2x^3dx^1 \wedge dx^3 \wedge dx^4 - x^1x^2x^4dx^2 \wedge dx^3 \wedge dx^4.\end{aligned}$$

If $V = V^a\partial_a$ then

$$\begin{aligned}V \lrcorner \Omega &= V^1dx^2 \wedge dx^3 \wedge dx^4 - V^2dx^1 \wedge dx^3 \wedge dx^4\\ &\quad + V^3dx^1 \wedge dx^2 \wedge dx^4 - V^4dx^1 \wedge dx^2 \wedge dx^3.\end{aligned}$$

Thus in order that $V \lrcorner \Omega = \alpha \wedge \beta$ we must have

$$\begin{aligned}V^1 &= -x^1x^2x^4\\ V^2 &= -x^1x^2x^3\\ V^3 &= -x^2x^3x^4\\ V^4 &= -x^1x^3x^4,\end{aligned}$$

and so

$$V = -x^1x^2x^4\partial_1 - x^1x^2x^3\partial_2 - x^2x^3x^4\partial_3 - x^1x^3x^4\partial_4.$$

(e) [5 marks]
The divergence $\mathrm{div}_\Omega V$ is given by

$$\mathcal{L}_V\Omega = (\mathrm{div}_\Omega V)\Omega.$$

Now

$$\begin{aligned}\mathcal{L}_V\Omega &= V\left(1 + (x^1)^2 + (x^2)^2 + (x^3)^2\right)dx^1 \wedge dx^2 \wedge dx^3\\ &\quad + \left(1 + (x^1)^2 + (x^2)^2 + (x^3)^2\right)\\ &\quad \times \left(dV^1 \wedge dx^2 \wedge dx^3 + dx^1 \wedge dV^2 \wedge dx^3 + dx^1 \wedge dx^2 \wedge dV^3\right)\\ &= \left(2(x^1)^2 + 2(x^2)^2 + 2(x^3)^2\right)dx^1 \wedge dx^2 \wedge dx^3\\ &\quad + \left(1 + (x^1)^2 + (x^2)^2 + (x^3)^2\right)3dx^1 \wedge dx^2 \wedge dx^3\\ &= \left(3 + 5((x^1)^2 + (x^2)^2 + (x^3)^2)\right)dx^1 \wedge dx^2 \wedge dx^3.\end{aligned}$$

Thus

$$\mathrm{div}_\Omega V = \frac{3 + 5((x^1)^2 + (x^2)^2 + (x^3)^2)}{1 + (x^1)^2 + (x^2)^2 + (x^3)^2}.$$