$$
\begin{aligned}
&+ d(V \lrcorner \alpha) \wedge \beta + (-1)^{q-1}(V \lrcorner \alpha) \wedge d\beta \\
&+ (-1)^q d\alpha \wedge (V \lrcorner \beta) + (-1)^{2q}\alpha \wedge d(V \lrcorner \beta) \\
= \ & (V \lrcorner d\alpha + d(V \lrcorner \alpha)) \wedge \beta + \alpha \wedge (V \lrcorner d\beta + d(V \lrcorner \beta)) \\
&+ \left((-1)^{q+1} + (-1)^q\right) d\alpha \wedge (V \lrcorner \beta) \\
&+ \left((-1)^q + (-1)^{q-1}\right) (V \lrcorner \alpha) \wedge d\beta \\
= \ & (\tilde{\mathcal{L}}_V \alpha) \wedge \beta + \alpha \wedge (\tilde{\mathcal{L}}_V \beta).
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathcal{L}}_V(d\alpha) &= V \lrcorner d^2\alpha + d(V \lrcorner d\alpha) = d(V \lrcorner d\alpha) \\
&= d\left((V \lrcorner d\alpha) + d(V \lrcorner \alpha)\right) \\
&= d(\tilde{\mathcal{L}}_V \alpha).
\end{aligned}
$$

Question 6

(i) [*5 marks*]
The connection ∇ is symmetric if its torsion is zero, that is, if

$$\nabla_U V - \nabla_V U = [U, V]$$

for any pair of vector fields U, V. The connection coefficients Γ^a_{bc} are defined by

$$\nabla_{\partial_c} \partial_b = \Gamma^a_{bc} \partial_a$$

and the connection is symmetric if and only if

$$\Gamma^a_{cb} = \Gamma^a_{bc}.$$

(ii) [*7 marks*]

$$R(U,V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W.$$

$$
\begin{aligned}
R(U,V)fW &= \nabla_U \nabla_V (fW) - \nabla_V \nabla_U (fW) - \nabla_{[U,V]}(fW) \\
&= \nabla_U (f \nabla_V W + (Vf)W) - \nabla_V (f \nabla_U W + (Uf)W) \\
&\quad - \nabla_{[U,V]}(fW) \\
&= f \nabla_U \nabla_V W + (Uf) \nabla_V W + (Vf) \nabla_U W + U(Vf)W \\
&\quad - f \nabla_V \nabla_U W - (Vf) \nabla_U W - (Uf) \nabla_V W - V(Uf)W \\
&\quad - f \nabla_{[U,V]} W - ([U,V]f)W \\
&= f(\nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W) \\
&= f R(U,V)W.
\end{aligned}
$$

To show that R is a tensor field it is necessary to confirm, in addition, that it is additive in each vector argument, and that for every function f

$$R(fU, V)W = R(U, fV)W = fR(U,V)W.$$