Thus

$$T = \tilde{T} = T^a_b e_a \otimes \theta^b$$

and so T may be expressed as a linear combination of the $e_a \otimes \theta^b$. The dimension of T is thus $\dim \mathcal{V} \times \dim \mathcal{V}^* = (\dim \mathcal{V})^2$.

(iv) [*6 marks*]
The map $\bar{\lambda}$ is bilinear:

$$\begin{aligned}\bar{\lambda}(k_1\beta_1 + k_2\beta_2, w) &= \langle \lambda(w), k_1\beta_1 + k_2\beta_2 \rangle \\ &= k_1\langle \lambda(w), \beta_1 \rangle + k_2\langle \lambda(w), \beta_2 \rangle \\ &= k_1\bar{\lambda}(\beta_1, w) + k_2\bar{\lambda}(\beta_2, w)\end{aligned}$$

and

$$\begin{aligned}\bar{\lambda}(\beta, k_1w_1 + k_2w_2) &= \langle \lambda(k_1w_1 + k_2w_2), \beta \rangle \\ &= \langle k_1\lambda(w_1) + k_2\lambda(w_2), \beta \rangle \\ &= k_1\langle \lambda(w_1), \beta \rangle + k_2\langle \lambda(w_2), \beta \rangle \\ &= k_1\bar{\lambda}(\beta, w_1) + k_2\bar{\lambda}(\beta, w_2).\end{aligned}$$

The map $\lambda \mapsto \bar{\lambda}$ is linear. If $\bar{\lambda} = 0$ then $\langle \lambda(w), \beta \rangle = 0$ for all β, w, so $\lambda(w) = 0$ for all w, and so $\lambda = 0$. Thus the map $\lambda \mapsto \bar{\lambda}$ is injective. The space of linear maps $\mathcal{V} \to \mathcal{V}$ has dimension $(\dim \mathcal{V})^2$, so the map $\lambda \mapsto \bar{\lambda}$, being an injective linear map between spaces of the same dimension, is an isomorphism. Alternatively, if $\bar{\lambda} = \lambda^a_b e_a \otimes \theta^b$ then $\bar{\lambda}$ is the image of the linear map λ whose matrix with respect to the basis $\{e_a\}$ is (λ^a_b), so the map $\lambda \mapsto \bar{\lambda}$ is also surjective.