Question 8

(i) [*4 marks*]
The curve $t \mapsto \exp tX$ is the integral curve of the left-invariant vector field X through e, so its tangent vector there (at $t = 0$) is X_e. Now $I_* X_e$ is the tangent vector at $t = 0$ to the curve $t \mapsto I(\exp tX)$. Note that $I(e) = e$, so $I_* X_e$ is also a tangent vector at e. But

$$I(\exp tX) = (\exp tX)^{-1} = \exp(-tX),$$

and the tangent vector to the curve $t \mapsto \exp(-tX)$ at $t = 0$ is $-X_e$. Thus

$$I_* X_e = -X_e.$$

This is equivalent to the given result when $\mathcal{G}$ is identified with $T_e G$.

(ii) [*6 marks*]
Left translation by $g \in G$ is the map $L_g : G \to G$ given by $L_g h = gh$. A p-form ω on G is left-invariant if, for every $g \in G$,

$$L_g^* \omega = \omega.$$

~~Ha~~ the case that L_g takes p-forms on G to ~~p~~forms on G.
OT ω, $L_g^*\omega$ are the same covector field.

This means that

$$L_g^*(\omega_g) = \omega_e,$$

since $L_g e = g$ and L_g^* maps contragrediently; thus

$$\omega_g = L_{g^{-1}}^* \omega_e.$$

Thus a left-invariant form ω is determined completely by its value at e. Now sums and constant multiples of left-invariant p-forms are also left-invariant p-forms; the set of left-invariant p-forms therefore constitutes a vector space. It follows from the comments above that evaluation at e is an isomorphism of the space of left-invariant p-forms with the p-fold exterior product of $T_e^* G$, which is finite-dimensional; thus the set of left-invariant p-forms is a finite-dimensional vector space.

(iii) [*6 marks*]
Note that

$$I(R_g h) = (hg)^{-1} = g^{-1} h^{-1} = L_{g^{-1}} I(h),$$

so

$$I \circ R_g = L_{g^{-1}} \circ I.$$

If ω is left-invariant then for every $g \in G$

$$\begin{aligned} R_g^* I^* \omega &= (I \circ R_g)^* \omega \\ &= (L_{g^{-1}} \circ I)^* \omega = I^* L_{g^{-1}}^* \omega \\ &= I^* \omega, \end{aligned}$$

so $I^* \omega$ is right-invariant. If ω is a left-invariant p-form and $X_1, X_2, \ldots, X_p$ are any p vectors at e then

$$\begin{aligned} (I^*\omega)_e(X_1, X_2, \ldots, X_p) &= \omega_e(I_* X_1, I_* X_2, \ldots, I_* X_p) \\ &= \omega_e(-X_1, -X_2, \ldots, -X_p) \\ &= (-1)^p \omega_e(X_1, X_2, \ldots, X_p) \end{aligned}$$