**Question 1**

Throughout this question $(x^1, x^2, \ldots, x^m)$ are local, and sometimes global, coordinates on a differentiable manifold $\mathcal{M}$ of dimension $m$ (either 2, 3 or 4).

(a) A smooth map $\phi$ from a 2-dimensional manifold $\mathcal{M}$ to a 3-dimensional manifold is given in terms of local coordinates by

$$\phi(x^1, x^2) = (x^1 + x^2, x^1 - x^2, x^1 x^2).$$

Specify the image of the tangent space to $\mathcal{M}$, at the point whose coordinates are $(0, 0)$, under the induced map $\phi_*$.

(b) A one-parameter family of maps of a 2-dimensional affine space is given in terms of affine coordinates by

$$\phi_t(x^1, x^2) = (x^1 + t, x^2 + (1 - 2x^1)t - t^2).$$

Show that the one-parameter family is in fact a one-parameter group, and find its infinitesimal generator.

(c) A vector field $V$ and a 1-form $\alpha$ on a 3-dimensional manifold are given in terms of local coordinates by

$$V = (x^2 - x^3)\partial_1 + (x^3 - x^1)\partial_2 + (x^1 - x^2)\partial_3$$

$$\alpha = x^1 dx^1 + x^2 dx^2 + x^3 dx^3.$$

Compute $\mathcal{L}_V \alpha$.

(d) Forms $\alpha$, $\beta$ and $\Omega$ on a 4-dimensional manifold are given in terms of local coordinates by

$$\alpha = x^3 dx^1 + x^4 dx^2 - x^1 dx^3 - x^2 dx^4$$

$$\beta = x^2 x^3 dx^1 \wedge dx^4 + x^1 x^4 dx^2 \wedge dx^3$$

$$\Omega = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4.$$

Find a vector field $V$ such that

$$V \lrcorner \Omega = \alpha \wedge \beta.$$

(e) Find the divergence of the vector field $V$, on a 3-dimensional manifold, given in terms of local coordinates by

$$V = x^1\partial_1 + x^2\partial_2 + x^3\partial_3,$$

with respect to the volume form

$$\Omega = (1 + (x^1)^2 + (x^2)^2 + (x^3)^2) dx^1 \wedge dx^2 \wedge dx^3.$$

(f) Show that the 2-dimensional distribution, on a 3-dimensional affine space, which is spanned by the vector fields $V_1$, $V_2$ given in terms of affine coordinates by

$$V_1 = x^2\partial_1 + e^{x^1}\partial_2$$

$$V_2 = x^3\partial_1 + e^{x^1}\partial_3,$$

is integrable.

(g) The metric on a 2-dimensional Riemannian manifold is

$$(dx^1)^2 + 2\cos\phi\, dx^1 dx^2 + (dx^2)^2,$$

where $\phi$ is some function of the coordinates (such that $0 < \phi < \frac{\pi}{2}$ everywhere). Find the vector fields obtained by lifting the indices on the coordinate 1-forms $dx^1$ and $dx^2$ using this metric.

(h) A connection on a 2-dimensional manifold is defined, in terms of local coordinates, by giving its connection 1-forms $\omega^a_b$ associated with the local basis of vectors fields $\{U_a\}$ where

$$U_1 = \tfrac{1}{2}(\partial_1 + e^{-x^1}\partial_2) \qquad U_2 = \tfrac{1}{2}(\partial_1 - e^{-x^1}\partial_2).$$