Question 4

(i) [*4 marks*]
The dimensions of $T_x\mathcal{M}$ and $T_x^*\mathcal{M}$ are the same, so it is enough to show that the map $v \mapsto v \lrcorner \Omega_x$ is injective. So suppose that, for some v, $v \lrcorner \Omega_x = 0$. If v were non-zero one could construct a basis $\{e_1, e_2\}$ for $T_x\mathcal{M}$ with $e_1 = v$: then $\Omega_x(e_1, e_2) = 0$; but this would mean that $\Omega_x = 0$ whereas Ω is nowhere vanishing. Thus $v = 0$ and $v \mapsto v \lrcorner \Omega_x$ is injective.

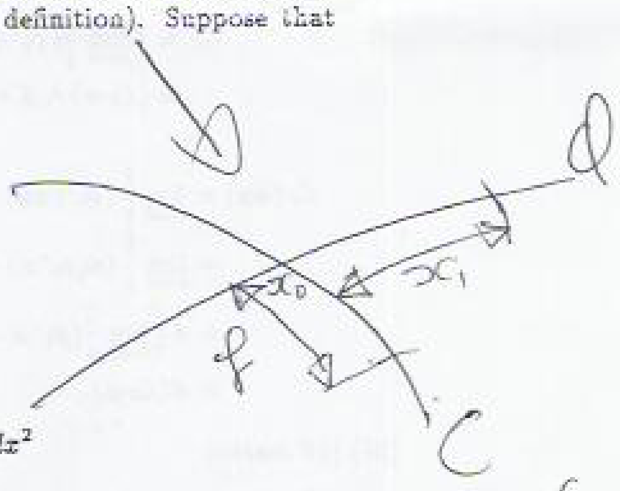
(ii) [*5 marks*]
Since $v \mapsto v \lrcorner \Omega_x$ is an isomorphism, for every $x \in \mathcal{M}$ there is a unique vector V_x such that $V_x \lrcorner \Omega_x = df|_x$. The resulting vector field V is smooth because Ω and df are, and is uniquely determined; $V_x \neq 0$ so long as $df|_x \neq 0$; and

$$Vf = \langle V, df \rangle = \langle V, V \lrcorner \Omega \rangle = \Omega(V, V) = 0.$$

(iii) [*6 marks*]
Since $Vf = 0$ the integral curves of V coincide (as subsets of $\mathcal{M}$) with the level surfaces (actually curves, since $\mathcal{M}$ is 2-dimensional) of f. Choose a curve C through x_0 transverse to the integral curves of V. Define coordinates (x^1, x^2) of any point x sufficiently close to x_0 as follows: $x^1(x)$ is the parameter distance of x from C along the integral curve of V on which x lies; $x^2(x) = f(x)$. These are good coordinates on a neighbourhood of x_0, since dx^1 and dx^2 are linearly independent ($Vx^1 = 1$ while $Vx^2 = 0$) and therefore the Jacobian of the coordinate transformation from any pre-existing coordinates to (x^1, x^2) is non-singular; and the rule given in the definition of (x^1, x^2) assigns unique coordinates to each point sufficiently close to x_0. With respect to these coordinates $V = \partial/\partial x^1$ (because x^1 is the parameter along the integral curves of V) and $f = x^2$ (by definition). Suppose that $\Omega = a\,dx^1 \wedge dx^2$ for some function a. Now

ie f constant on integral curves of V

$$V \lrcorner \Omega = \frac{\partial}{\partial x^1} \lrcorner (a\,dx^1 \wedge dx^2) = a\,dx^2.$$



Vf = 0 on C.

But $V \lrcorner \Omega = df = dx^2$. Thus $a = 1$ and

$$\Omega = dx^1 \wedge dx^2.$$

(iv) [*5 marks*]
If $W = W^1\partial_1 + W^2\partial_2$ then

$$\begin{aligned}(W^1\partial_1 + W^2\partial_2) \lrcorner \Omega = dh &= \frac{\partial h^x}{\partial x^1}dx^1 + \frac{\partial h^x}{\partial x^2}dx^2 \\ &= W^1 dx^2 - W^2 dx^1,\end{aligned}$$

and so

$$W^1 = \frac{\partial h^x}{\partial x^2} \qquad W^2 = -\frac{\partial h^x}{\partial x^1}.$$

The equations for an integral curve σ of W, in terms of the coordinate presentation of σ with respect to the given coordinates, are

$$\frac{d\sigma^1}{dt} = \frac{\partial h^x}{\partial x^2}(\sigma^1, \sigma^2) \qquad \frac{d\sigma^2}{dt} = -\frac{\partial h^x}{\partial x^1}(\sigma^1, \sigma^2).$$