(f) [*4 marks*]
The condition for the distribution to be integrable is that $[V_1, V_2]$ should be a linear combination of V_1 and V_2. But

$$\begin{aligned}[V_1, V_2] &= \left[x^2\partial_1 + e^{x^1}\partial_2, x^3\partial_1 + e^{x^1}\partial_3\right] \\ &= x^2e^{x^1}\partial_3 - x^3e^{x^1}\partial_2 = x^2V_2 - x^3V_1,\end{aligned}$$

which is indeed a linear combination of V_1 and V_2.

(g) [*4 marks*]
The vector field obtained by lifting the index on the 1-form $\alpha_a dx^a$ is $g^{ab}\alpha_a\partial_b$, where (g^{ab}) is the matrix inverse to the matrix (g_{ab}) of coefficients of the metric. The required vector fields are therefore $g^{1b}\partial_b$ and $g^{2b}\partial_b$. The matrix inverse to

$$\begin{pmatrix} 1 & \cos\phi \\ \cos\phi & 1 \end{pmatrix}$$

is

$$\operatorname{cosec}^2\phi \begin{pmatrix} 1 & -\cos\phi \\ -\cos\phi & 1 \end{pmatrix}$$

so the required vector fields are

$$\operatorname{cosec}^2\phi\,(\partial_1 - \cos\phi\,\partial_2) \qquad \text{and} \qquad \operatorname{cosec}^2\phi\,(-\cos\phi\,\partial_1 + \partial_2).$$

(h) [*8 marks*]

$$\begin{aligned} d\theta^1 + \omega^1_b \wedge \theta^b &= e^{x^1} dx^1 \wedge dx^2 + \left(e^{x^1} dx^2\right) \wedge \left(dx^1 - e^{x^1} dx^2\right) = 0 \\ d\theta^2 + \omega^2_b \wedge \theta^b &= -e^{x^1} dx^1 \wedge dx^2 + \left(-e^{x^1} dx^2\right) \wedge \left(dx^1 + e^{x^1} dx^2\right) = 0. \end{aligned}$$

The torsion 1-forms therefore vanish. The covariant derivatives of the basis vector fields are given by

$$\nabla_V U_a = \langle V, \omega^b_a \rangle U_b.$$

Now

$$\partial_2 = e^{x^1}(U_1 + U_2)$$

and so

$$\begin{aligned}\nabla_{\partial_1}\partial_2 &= \nabla_{\partial_1}\left(e^{x^1}(U_1 + U_2)\right) \\ &= e^{x^1}(U_1 + U_2) + e^{x^1}\left(\nabla_{\partial_1}U_1 + \nabla_{\partial_1}U_2\right).\end{aligned}$$

But $\langle \partial_1, \omega^b_a \rangle = 0$ and so

$$\nabla_{\partial_1}\partial_2 = e^{x^1}(U_1 + U_2) = e^{x^1}\partial_1.$$

Since there is no torsion, and $[\partial_1, \partial_2] = 0$,

$$\nabla_{\partial_2}\partial_1 = \nabla_{\partial_1}\partial_2 = e^{x^1}\partial_1.$$