Question 8

Let G be a Lie group, $\mathcal{G}$ its Lie algebra. Denote by $I : G \to G$ the inversion map $g \mapsto g^{-1}$.

(i) By using the facts that for any $X \in \mathcal{G}$,

$$(\exp tX)^{-1} = \exp(-tX),$$

and that $\mathcal{G}$ may be regarded as the tangent space to G at the identity e, show that the map $I_* : T_eG \to T_eG$ is given by

$$I_*X = -X.$$

(ii) Explain what a left-invariant p-form on G is. Show that a left-invariant form is uniquely determined by its value at e, and deduce that the set of left-invariant forms of a given degree is a finite-dimensional vector space.

A p-form ω on G is called right-invariant if

$$R_g^*\omega = \omega,$$

where R_g represents right translation by $g \in G$.

(iii) Show that if ω is left-invariant then $I^*\omega$ is right-invariant. Show that if ω is a left-invariant p-form and $\tilde{\omega}$ is the right-invariant p-form such that $\tilde{\omega}_e = \omega_e$ then

$$\tilde{\omega} = (-1)^p I^*\omega.$$

(iv) Let $\{\alpha^a\}$ be a basis for the left-invariant 1-forms on G; the 1-forms α^a then satisfy the Maurer-Cartan equations

$$d\alpha^a = -\tfrac{1}{2}C^a_{bc}\alpha^b \wedge \alpha^c$$

where the C^a_{bc} are the structure constants of $\mathcal{G}$. Let $\{\tilde{\alpha}^a\}$ be the basis for the right-invariant 1-forms such that $\tilde{\alpha}^a$ coincides with α^a at the identity. Show that

$$d\tilde{\alpha}^a = \tfrac{1}{2}C^a_{bc}\tilde{\alpha}^b \wedge \tilde{\alpha}^c.$$

[END OF QUESTION PAPER]