Question 3

(i) [*6 marks*]
A chart on $\mathcal{M}$ is a pair $(\mathcal{P}, \psi)$ consisting of an open subset $\mathcal{P}$ of $\mathcal{M}$ and a map $\psi: \mathcal{P} \to \mathbf{R}^m$ which is a homeomorphism of $\mathcal{P}$ onto an open subset of $\mathbf{R}^m$. Charts $(\mathcal{P}_1, \psi_1)$ and $(\mathcal{P}_2, \psi_2)$ are smoothly related if either $\mathcal{P}_1 \cap \mathcal{P}_2$ is empty, or it is non-empty and the maps $\psi_2 \circ \psi_1^{-1}$ and $\psi_1 \circ \psi_2^{-1}$ are C^∞ maps of open subsets of $\mathbf{R}^m$. A smooth atlas for $\mathcal{M}$ is a collection of charts $\{(\mathcal{P}_\alpha, \psi_\alpha)\}$, each two of which are smoothly related, and such that the $\mathcal{P}_\alpha$ cover $\mathcal{M}$.

(ii) [*10 marks*]
For charts $(\mathcal{P}_1, \psi_1)$ and $(\mathcal{P}_2, \psi_2)$ on $\mathcal{M}$ with $\mathcal{P}_1 \cap \mathcal{P}_2 \neq \emptyset$ the components (u_1^a) and (u_2^a) of the same tangent vector at $x \in \mathcal{P}_1 \cap \mathcal{P}_2$ are related by

$$u_2^a = \frac{\partial x_2^a}{\partial x_1^b} u_1^b$$

where the Jacobian matrix of the coordinate transformation is evaluated at $\psi_1(x)$, in other words in terms of the coordinates of x with respect to the first chart. Thus the map $\tilde{\psi}_2 \circ \tilde{\psi}_1^{-1}$ is given by

$$(x_1^a, u_1^a) \mapsto \left(x_2^a, \frac{\partial x_2^a}{\partial x_1^b} u_1^b \right)$$

where $(x_1^a) \mapsto (x_2^a)$ is the smooth map $\psi_2 \circ \psi_1^{-1}$ of open subsets of $\mathbf{R}^n$, $n = \dim \mathcal{N}$. Since the map $(x_1^a) \mapsto (x_2^a)$ is smooth, so is its Jacobian matrix $(\partial x_2^a / \partial x_1^b)$, and therefore $(u_1^a) \mapsto ((\partial x_2^a / \partial x_1^b) u_1^b)$ is smooth in both x_1^a and u_1^a. It follows that the coordinate transformation $(x_1^a, u_1^a) \mapsto (x_2^a, u_2^a)$ is smooth. Its inverse, which is obtained by interchanging 1 and 2 throughout, is also smooth. Each $(\tilde{\mathcal{P}}_\alpha, \tilde{\psi}_\alpha)$ is a chart on $\mathcal{M}$. These charts are pairwise smoothly related, as has just been shown. The $\tilde{\mathcal{P}}_\alpha$ cover $\mathcal{M}$: for given any $u \in \mathcal{M}$, the point of definition of u lies in some $\mathcal{P}_\alpha$, and then $u \in \tilde{\mathcal{P}}_\alpha$. So $\{(\tilde{\mathcal{P}}_\alpha, \tilde{\psi}_\alpha)\}$ is a smooth atlas for $\mathcal{M}$; and $\mathcal{M}$, having a smooth atlas, is a differentiable manifold (with the differentiable structure obtained by adding all charts smoothly related to those in the given atlas). Its dimension is $2n = 2 \dim \mathcal{N}$.

(iii) [*4 marks*]
In terms of coordinates (x^a, u^a) on $\mathcal{M}$, π is given by $(x^a, u^a) \mapsto (x^a)$. It is clearly smooth. Its Jacobian matrix in this coordinate system is the $2n \times n$ matrix

$$\left(I_n, 0_n \right)$$

where I_n is the $n \times n$ unit matrix and 0_n is an $n \times n$ matrix of zeros. Thus the rank of π is n, which is the dimension of its codomain $\mathcal{N}$, and so π is a submersion. It is also surjective, for given any $x \in \mathcal{N}$ there is $u \in \mathcal{M}$ such that $\pi(u) = x$, namely any tangent vector at x.