Question 4

Let Ω be a nowhere vanishing 2-form on a 2-dimensional manifold $\mathcal{M}$.

(i) Show that for each point $x \in \mathcal{M}$ the map $T_x\mathcal{M} \to T_x^*\mathcal{M}$ by $v \mapsto v \lrcorner \Omega_x$ is an isomorphism.

(ii) Let f be a smooth function on $\mathcal{M}$ such that df is non-vanishing in a neighbourhood of some given point x_0. Show that there is a unique smooth vector field V on $\mathcal{M}$ such that $V \lrcorner \Omega = df$; that V is non-vanishing in the same neighbourhood of x_0; and that $Vf = 0$.

(iii) Deduce that coordinates (x^1, x^2) may be found in a (possibly smaller) neighbourhood of x_0 such that $V = \partial/\partial x^1$ and f is the coordinate function x^2. Show that with respect to these coordinates

$$\Omega = dx^1 \wedge dx^2.$$

(iv) Show that for any smooth function h on $\mathcal{M}$ each intergal curve σ of the vector field W defined by $W \lrcorner \Omega = dh$, when expressed in terms of the coordinates found in part (iii), is a solution of the differential equations

$$\frac{d\sigma^1}{dt} = \frac{\partial h^x}{\partial x^2} \qquad \frac{d\sigma^2}{dt} = -\frac{\partial h^x}{\partial x^1}.$$

Question 5

(i) Let V be a vector field on a differential manifold, with flow ϕ_t. Define the Lie derivative $\mathcal{L}_V\omega$ of a p-form ω with respect to V, as a limit involving the pull-back map ϕ_t^*.

(ii) Assuming that for any smooth map ϕ, any smooth function f and any forms α and β

$$\phi^*(f\alpha) = (f \circ \phi)(\phi^*\alpha)$$
$$\phi^*(\alpha \wedge \beta) = (\phi^*\alpha) \wedge (\phi^*\beta)$$
$$\phi^*(d\alpha) = d(\phi^*\alpha)$$

show that

$$\mathcal{L}_V(f\alpha) = f\mathcal{L}_V\alpha + (Vf)\alpha$$
$$\mathcal{L}_V(\alpha \wedge \beta) = (\mathcal{L}_V\alpha) \wedge \beta + \alpha \wedge (\mathcal{L}_V\beta)$$
$$\mathcal{L}_V(d\alpha) = d(\mathcal{L}_V\alpha).$$

(iii) Let $\tilde{\mathcal{L}}_V$ denote the map of forms

$$\omega \mapsto V \lrcorner d\omega + d(V \lrcorner \omega).$$

Without assuming that $\tilde{\mathcal{L}}_V = \mathcal{L}_V$, show that

$$\tilde{\mathcal{L}}_V(f\alpha) = f\tilde{\mathcal{L}}_V\alpha + (Vf)\alpha$$
$$\tilde{\mathcal{L}}_V(\alpha \wedge \beta) = (\tilde{\mathcal{L}}_V\alpha) \wedge \beta + \alpha \wedge (\tilde{\mathcal{L}}_V\beta)$$
$$\tilde{\mathcal{L}}_V(d\alpha) = d(\tilde{\mathcal{L}}_V\alpha).$$