Question 1

(a) Let $\ell_k(x)$, $k = 0, 1, 2, 3$ denote the Lagrange functions for the interpolation points $\{0, 1, 2, 3\}$ in $[0,3]$. Determine

$$\max_{0 \le x \le 3} \sum_{k=0}^{3} |\ell_k(x)|.$$

(b) For $f \in C[0, 3]$, let Xf denote the element of $\mathcal{P}_3$ which interpolates f at $\{0, 1, 2, 3\}$, and suppose that

$$Xf(\alpha) = f(\alpha) + 0.2,$$

for some α in $[0,3]$. Deduce from part (a) that

$$\|f - p\|_\infty \ge 0.076, \quad p \in \mathcal{P}_3.$$

(c) Briefly explain how you might use the value of α in part (b) to improve this lower estimate for $\|f - p\|_\infty, p \in \mathcal{P}_3$.

Question 2

A function f in $C^{(3)}[0, 1]$ takes the values

$$f(0) = 0, \quad f(0.2) = 0.6, \quad f(1) = 1.$$

(a) Determine Newton's formula for the polynomial p in $\mathcal{P}_2$ which interpolates the above values of f.

(b) Deduce from part (a) that

$$\|f^{(2)}\|_\infty \ge 5,$$

and prove that this estimate is best possible.

(c) Suppose now that

$$f(0.5) = p(0.5) - 0.1,$$

where p is the polynomial in part (a). Deduce that

$$\|f^{(3)}\|_\infty \ge 8,$$

and prove that this estimate is best possible.

Question 3

Let $f(x) = \sqrt{x}$, $0 \le x \le 1$, and let B_n, $n = 1, 2, \ldots$, denote the Bernstein operator.

(a) Evaluate each of the polynomials $B_1 f$, $B_2 f$, $B_4 f$ at $x = \frac{1}{2}$.

(b) Evaluate $B_n f(0)$ and $B_n f(1)$, for $n = 1, 2, \ldots$.

(c) Prove that

$$(B_n f)'(0) \longrightarrow \infty \quad \text{as } n \longrightarrow \infty.$$

Question 4

(a) Use the fact that 'the best minimax approximation from $\mathcal{P}_n$ to an even function is even' to determine the best minimax approximation from $\mathcal{P}_3$ to $f(x) = |x|^3$, $-1 \le x \le 1$.

(b) Consider the problem of finding the best minimax approximation p^* from $\mathcal{P}_1$ to $f(x) = x^3 - x^2$ on $[-1, 1]$. Carry out *two* iterations of the one-point exchange algorithm with initial reference $\{-1, 0, 1\}$ and state what information you obtain about $\|f - p^*\|_\infty$.