(b) Using Theorem 11.2, we have

$$p^*(x) = \frac{(\varphi_0, f)}{\|\varphi_0\|^2}\varphi_0(x) + \frac{(\varphi_1, f)}{\|\varphi_1\|^2}\varphi_1(x) + \frac{(\varphi_2, f)}{\|\varphi_2\|^2}\varphi_2(x).$$

Now

$$(\varphi_0, f) = \int_{-1}^{1} x^2|x|\,dx = 2\int_0^1 x^3\,dx = 1/2,$$

$$(\varphi_1, f) = \int_{-1}^{1} x^2.x.|x|\,dx = \int_{-1}^{1} x^3|x|dx = 0,$$

$$\begin{aligned}(\varphi_2, f) &= \int_{-1}^{1} x^2(x^2 - 3/5)|x|\,dx = 2\int_0^1 (x^5 - 3x^3/5)\,dx \\ &= 2\left[\frac{x^6}{6} - \frac{3x^4}{20}\right]_0^1 = \frac{1}{30},\end{aligned}$$

and

$$\begin{aligned}\|\varphi_2\|^2 &= \int_{-1}^{1} x^2(x^2 - 3/5)^2\,dx = \int_{-1}^{1}\left(x^6 - \frac{6}{5}x^4 + \frac{9}{25}x^2\right)dx \\ &= 2\left[\frac{x^7}{7} - \frac{6}{25}x^5 + \frac{9}{75}x^3\right]_0^1 = \frac{8}{175}.\end{aligned}$$

Hence

$$\begin{aligned}p^*(x) &= \frac{1/2}{2/3}\varphi_0(x) + \frac{1/30}{8/175}\varphi_2(x) \\ &= \frac{3}{4} + \frac{35}{48}\left(x^2 - \frac{3}{5}\right) \\ &= \frac{5}{16} + \frac{35}{48}x^2.\end{aligned}$$

(c) We wish to apply Theorem 12.3 with $k = 1$ and $w(x) = x^2$ on $[-1, 1]$. The zeros of $\varphi_2(x) = x^2 - 3/5$ are $x_0 = -\sqrt{3/5}$ and $x_1 = \sqrt{3/5}$ and the coefficients c_0, c_1 are given by

$$\begin{aligned}c_0 &= \int_{-1}^{1} x^2 l_0(x)\,dx = \int_{-1}^{1} x^2\left(\frac{x - x_1}{x_0 - x_1}\right)dx \\ &= -\frac{1}{2\sqrt{3/5}}\int_{-1}^{1}(x^3 - x_1x^2)\,dx = \left(\frac{\sqrt{3/5}}{2\sqrt{3/5}}\right)2/3 = 1/3,\end{aligned}$$

and

$$\begin{aligned}c_1 &= \int_{-1}^{1} x^2\ell_1(x)\,dx = \int_{-1}^{1} x^2\left(\frac{x - x_0}{x_1 - x_0}\right)dx \\ &= \frac{1}{2\sqrt{3/5}}\int_{-1}^{1}(x^3 - x_0x^2)\,dx = \left(\frac{\sqrt{3/5}}{2\sqrt{3/5}}\right)2/3 = 1/3.\end{aligned}$$

Hence, by Theorem 12.3,

$$\int_{-1}^{1} x^2 f(x)\,dx \approx \frac{1}{3}f\left(-\sqrt{3/5}\right) + \frac{1}{3}f\left(\sqrt{3/5}\right)$$

is exact for cubics $(2k + 1 = 3)$.