Also

$$\frac{1}{\pi}\int_{-\pi}^{\pi} |x|^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^2}{3},$$

and so, by Parseval's Theorem,

$$\frac{1}{2}\pi^2 + \sum_{j=1}^{\infty}\left(\frac{4}{(2j-1)^2\pi}\right)^2 = \frac{2\pi^2}{3}.$$

Hence

$$\sum_{j=1}^{\infty}\frac{1}{(2j-1)^4} = \frac{\pi^2}{16}.\pi^2\left(\frac{2}{3}-\frac{1}{2}\right) = \frac{\pi^4}{96}.$$

Question 7

(a) We use equation (18.13) with $m = 5$, $h = \frac{1}{2}$:

$$\begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} s'(x_2) \\ s'(x_3) \\ s'(x_4) \end{pmatrix} = \begin{pmatrix} 6(f(x_3)-f(x_1)) - s'(x_1) \\ 6(f(x_4)-f(x_2)) \\ 6(f(x_5)-f(x_3)) - s'(x_5) \end{pmatrix},$$

where

$$x_1 = -1, x_2 = -\frac{1}{2}, x_3 = 0, x_4 = \frac{1}{2}, x_5 = 1.$$

Since

$$f(-1) = 1, f\left(-\frac{1}{2}\right) = \frac{1}{2}, f(0) = 0, f\left(\frac{1}{2}\right) = \frac{1}{2}, f(1) = 1,$$

and

$$s'(-1) = -1, s'(1) = 1,$$

we have

$$\begin{aligned} 4s_2' + s_3' \qquad &= -5, && (1)\\ s_2' + 4s_3' + s_4' &= 0, && (2)\\ s_3' + 4s_4' &= 5. && (3) \end{aligned}$$

Now (1) and (3) give

$$4s_2' + 2s_3' + 4s_4' = 0,$$

and together with (2) this gives $s_3' = 0, s_2' = -5/4$, $s_4' = 5/4$. Using (18.4), (18.5), (18.6) it then follows that

$$s(x) = \begin{cases} 1-(x+1)+\frac{1}{2}(x+1)^2-(x+1)^3, & -1 \le x \le -\frac{1}{2}, \\ \frac{1}{2}-\frac{5}{4}(x+\frac{1}{2})-(x+\frac{1}{2})^2+3(x+\frac{1}{2})^3, & -\frac{1}{2} \le x \le 0 \\ \frac{7}{2}x^2-3x^3, & 0 \le x \le \frac{1}{2}, \\ \frac{1}{2}+\frac{5}{4}(x-\frac{1}{2})-(x-\frac{1}{2})^2+(x-\frac{1}{2})^3, & \frac{1}{2} \le x \le 1. \end{cases}$$

(b) Using (19.24) we have

$$B(x) = \sum_{j=0}^{3}\lambda_j(x-\xi_j)_+^2,$$

where $\xi_0 = -1, \xi_1 = 0, \xi_2 = 1, \xi_3 = 3$ and

$$\lambda_j = \prod_{\substack{i=0\\ i\neq j}}^{3}\frac{1}{(\xi_i - \xi_j)}.$$