(c) According to Theorem 4.2, the error function $e = f - p$ satisfies

$$e(x) = \frac{1}{3!}(x-0)(x-0.2)(x-1)f^{(3)}(\xi),$$

for some ξ in $[0,1]$. Applying this result with

$$e(\tfrac{1}{2}) = f(\tfrac{1}{2}) - p(\tfrac{1}{2}) = -0.1,$$

we obtain, for some ξ in $[0,1]$,

$$-0.1 = \frac{1}{6} \times 0.5 \times 0.3 \times (-0.5) f^{(3)}(\xi),$$

that is,

$$f^{(3)}(\xi) = 8 \Rightarrow \|f^{(3)}\|_\infty \geq 8,$$

as required.

To prove that this estimate is best possible, we construct a cubic q which interpolates the data and also satisfies

$$q(\tfrac{1}{2}) - p(\tfrac{1}{2}) = 0.1, \qquad \|q^{(3)}\|_\infty = 8.$$

In order that q interpolates the data, we require

$$q(x) - p(x) = \alpha x(x-0.2)(x-1);$$

moreover,

$$\alpha \times 0.5 \times 0.3 \times (-0.5) = 0.1 \Rightarrow \alpha = -4/3,$$

so that $q(\frac{1}{2}) - p(\frac{1}{2}) = 0.1$. Thus

$$q^{(3)}(x) = 6\alpha = -8,$$

and so $\|q^{(3)}\|_\infty = 8$, as required.

Question 3

(a) Using

$$B_n f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(k/n),$$

we have

$$B_1 f(x) = (1-x).0 + x.1 = x,$$

so that $B_1 f(\frac{1}{2}) = \frac{1}{2}$;

$$B_2 f(x) = (1-x)^2.0 + 2x(1-x).\frac{1}{\sqrt{2}} + x^2.1,$$

so that $B_2 f(\frac{1}{2}) = 1/(2\sqrt{2}) + 1/4 = 0.604$;

$$\begin{aligned}B_4 f(x) &= (1-x)^4.0 + 4x(1-x)^3.\frac{1}{2} + 6x^2(1-x)^2.\frac{1}{\sqrt{2}} \\ &\quad + 4x^3(1-x).\frac{\sqrt{3}}{2} + x^4.1 \\ &= 2x(1-x)^3 + 3\sqrt{2}x^2(1-x)^2 + 2\sqrt{3}x^3(1-x) + x^4,\end{aligned}$$

so that $B_4 f(\frac{1}{2}) = (2 + 3\sqrt{2} + 2\sqrt{3} + 1)/16 = 0.669$.

(b) For each n,

$$B_n f(0) = \binom{n}{0}.1.1 f(0) = 0,$$

because $0^k = 0$, for $k = 1, 2, \ldots, n$.

For each n,

$$B_n f(1) = \binom{n}{n} 1.1.f(1) = 1,$$

because $(1-1)^{n-k} = 0$, for $k = 0, 1, 2, \ldots, n-1$.