Now

$$\lambda_0 = \frac{1}{(0-(-1))(1-(-1))(3-(-1))} = \frac{1}{8},$$
$$\lambda_1 = \frac{1}{(-1-0)(1-0)(3-0)} = -\frac{1}{3},$$
$$\lambda_2 = \frac{1}{(-1-1)(0-1)(3-1)} = \frac{1}{4},$$
$$\lambda_3 = \frac{1}{(-1-3)(0-3)(1-3)} = -\frac{1}{24},$$

so that

$$B(x) = \frac{1}{8}(x+1)_+^2 - \frac{1}{3}(x)_+^2 + \frac{1}{4}(x-1)_+^2 - \frac{1}{24}(x-3)_+^2.$$

Hence, in piecewise polynomial form,

$$B(x) = \begin{cases} \frac{1}{8}(x+1)^2, & -1 \le x \le 0, \\ \frac{1}{8}(x+1)^2 - \frac{1}{3}x^2 = -\frac{5}{24}x^2 + \frac{1}{4}x + \frac{1}{8}, & 0 \le x \le 1, \\ \frac{1}{8}(x+1)^2 - \frac{1}{3}x^2 + \frac{1}{4}(x-1)^2 = \frac{1}{24}(x-3)^2, & 1 \le x \le 3. \end{cases}$$

Finally,

$$\begin{aligned}\int_{-1}^{3} B(x)\,dx &= \int_{-1}^{0} \frac{1}{8}(x+1)^2\,dx + \int_0^1 \left(-\frac{5}{24}x^2 + \frac{1}{4}x + \frac{1}{8}\right) dx + \int_1^3 \frac{1}{24}(x-3)^2\,dx \\ &= \left[\frac{1}{24}(x+1)^3\right]_{-1}^{0} + \left[-\frac{5}{72}x^3 + \frac{1}{8}x^2 + \frac{1}{8}x\right]_0^1 + \left[\frac{1}{72}(x-3)^3\right]_1^3 \\ &= \frac{1}{24} - \frac{5}{72} + \frac{1}{8} + \frac{1}{8} + \frac{8}{72} \\ &= \frac{1}{3},\end{aligned}$$

as expected (in view of (22.40)).

Question 8

(a) Let $p(x) = a + bx + cx^2$. Then

$$p(0) = f(0): \quad a = f(0),$$
$$p(1) = f(1): \quad a + b + c = f(1),$$
$$p'(\alpha) = f'(\alpha): \quad b + 2c\alpha = f'(\alpha).$$

Since $a = f(0)$,

$$b + c = f(1) - f(0),$$

so that

$$c = \frac{1}{2\alpha - 1}(f'(\alpha) - f(1) + f(0)), \qquad (\text{recall } \alpha \ne \tfrac{1}{2})$$
$$b = f(1) - f(0) - \frac{1}{2\alpha - 1}(f'(\alpha) - f(1) + f(0)).$$

Thus

$$\begin{aligned} p(\alpha) &= f(0) + \alpha\left(f(1) - f(0) - \frac{1}{2\alpha-1}(f'(\alpha) - f(1) + f(0))\right) \\ &\quad + \frac{\alpha^2}{2\alpha-1}(f'(\alpha) - f(1) + f(0)) \\ &= \left[1 - \alpha - \frac{\alpha}{2\alpha-1} + \frac{\alpha^2}{2\alpha-1}\right] f(0) + \left[\alpha + \frac{\alpha}{2\alpha-1} - \frac{\alpha^2}{2\alpha-1}\right] f(1) \\ &\qquad + \left[-\frac{\alpha}{2\alpha-1} + \frac{\alpha^2}{2\alpha-1}\right] f'(\alpha) \\ &= c_0 f(0) + c_1 f(1) + c_2 f'(\alpha), \end{aligned}$$

where

$$c_0 = -\frac{(\alpha-1)^2}{2\alpha-1}, c_1 = \frac{\alpha^2}{2\alpha-1}, c_2 = \frac{\alpha^2 - \alpha}{2\alpha - 1}.$$