(c) According to the second line of equation (6.32)

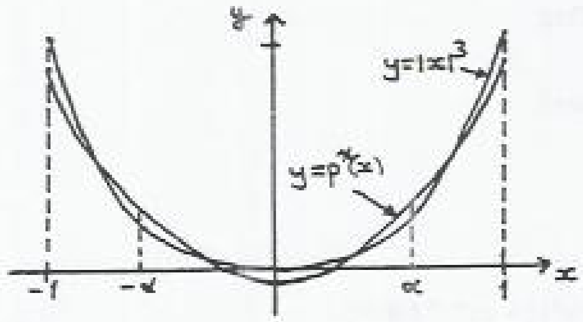
$$\begin{aligned}(B_{n+1}f)'(0) &= \sum_{k=0}^{n} \frac{(n+1)!}{k!(n-k)!} 0^k (1-0)^{n-k} \left\{ f\left(\frac{k+1}{n+1}\right) - f\left(\frac{k}{n+1}\right) \right\} \\ &= \frac{(n+1)!}{0!n!} \left\{ f\left(\frac{1}{n+1}\right) - f(0) \right\} \\ &= (n+1)\sqrt{\frac{1}{n+1}} \\ &= \sqrt{n+1}.\end{aligned}$$

(Alternatively: obtain $(B_n f)'(0) = \sqrt{n}$ directly.)

Since $\sqrt{n+1} \to \infty$ as $n \to \infty$, we deduce that $B_n'(0) \to \infty$ as $n \to \infty$.

Question 4

(a) Since $f(x) = |x|^3$ is even on $[-1,1]$, the b.m.a. p^* from $\mathcal{P}_3$ to f must be even by the given result, and hence of the form $p^*(x) = ax^2 + b$, where $a, b \in \mathcal{R}$. Moreover, by Theorem 7.2, there is an alternating set of length $3 + 2 = 5$, which must be symmetric about 0, and hence of the form $\{-1, -\alpha, 0, \alpha, 1\}$.



Hence

$$f(0) - p^*(0) = 0 - b = h, \tag{1}$$
$$f(\alpha) - p^*(\alpha) = \alpha^3 - a\alpha^2 - b = -h, \tag{2}$$
$$f(1) - p^*(1) = 1 - a - b = h, \tag{3}$$

and also

$$f'(\alpha) - p^{*\prime}(\alpha) = 3\alpha^2 - 2a\alpha = 0. \tag{4}$$

From (1) and (3) we find that $a = 1$ and then (4) gives

$$\alpha = 2a/3 = 2/3 (\text{since } \alpha \neq 0).$$

Hence from (2) and (1)

$$\left(\frac{2}{3}\right)^3 - \left(\frac{2}{3}\right)^2 - b = -h = b \Rightarrow b = -\frac{2}{27};$$

and so

$$p^*(x) = x^2 - \frac{2}{27}.$$

(b) Denote a typical element of $\mathcal{P}_1$ by $p(x) = a + bx$.
Iteration 1 $\{-1, 0, 1\}$

$$f(-1) - p(-1) = -2 - a + b = h, \tag{5}$$
$$f(0) - p(0) = 0 - a = -h, \tag{6}$$
$$f(1) - p(1) = 0 - a - b = h. \tag{7}$$

From (5) and (7)

$$-2 + 2b = 0 \Rightarrow b = 1,$$

so that $b = 1$. Hence, by (6) and (7), $h = -\frac{1}{2}, a = -\frac{1}{2}$.

Thus $p_1(x) = -\frac{1}{2} + x$, with $|h| = \frac{1}{2}$.