(b) Since

$$Xf(\alpha) - f(\alpha) = 0.2,$$

we deduce that

$$\|f - Xf\|_\infty \geq 0.2.$$

Now by part (a) and Theorem 4.2, $\|X\|_\infty = 1.631130309$.

Hence, by Theorem 3.1,

$$0.2 \leq \|f - Xf\|_\infty \leq (1 + \|X\|_\infty) \min_{p \in \mathcal{P}_3} \|f - p\|_\infty.$$

Thus, for $p \in \mathcal{P}_3$,

$$\|f - p\|_\infty \geq \frac{0.2}{1 + \|X\|_\infty} = \frac{0.2}{2.63113039} = 0.076,$$

as required.

(c) Using the proof of Theorem 3.1,

$$\begin{aligned} 0.2 &= |f(\alpha) - Xf(\alpha)| \\ &= |(f - p^*)(\alpha) - (Xf - p^*)(\alpha)| \\ &\leq \|f - p^*\|_\infty + |X(f - p^*)(\alpha)|, \end{aligned}$$

where p^* is the b.m.a. from $\mathcal{P}_3$ to f. The estimate

$$|X(f - p^*)(\alpha)| \leq \|X\|_\infty \|f - p^*\|_\infty,$$

which takes no account of α, gives Theorem 3.1. We may be able to get a better upper estimate for $|X(f - p^*)(\alpha)|$ by maximising $|Xg(\alpha)|$ for $g \in C[0,3]$, subject to $\|g\|_\infty$ being constant.

Question 2

(a) First calculate the divided differences:

x_i	$f(x_i)$	Order 1	Order 2
0	0		
		3	
0.2	0.6		$0.5 - 3 = -2.5$
		0.5	
1	1		

Hence Newton's formula (with interpolation points taken from left to right) is

$$p(x) = 3x - 2.5x(x - 0.2).$$

(b) Since $f[0, 0.2, 1] = -2.5$, we deduce from Theorem 5.1 that

$$-2.5 = f^{(2)}(\xi)/2! \Rightarrow f^{(2)}(\xi) = -5,$$

for some ξ in $[0,1]$. Hence $\|f^{(2)}\|_\infty \geq 5$, as required.

This estimate is best possible because the function p found above lies in $C^{(3)}[0,1]$, takes the required values at $0, 0.2, 1$ and satisfies

$$\|p^{(2)}\|_\infty = |-5| = 5.$$