M832 Solutions to the Specimen Examination Paper

Question 1

(a) First

$$\ell_0(x) = \frac{(x-1)(x-2)(x-3)}{(0-1)(0-2)(0-3)} = -\frac{1}{6}(x-1)(x-2)(x-3)$$
$$\ell_1(x) = \frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} = \frac{1}{2}x(x-2)(x-3)$$
$$\ell_2(x) = \frac{(x-0)(x-1)(x-3)}{(2-0)(2-1)(2-3)} = -\frac{1}{2}x(x-1)(x-3)$$
$$\ell_3(x) = \frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)} = \frac{1}{6}x(x-1)(x-2).$$

Now put

$$\varphi(x) = \sum_{k=0}^{3} |\ell_k(x)|.$$

On $[0,1]$, we have

$$\begin{aligned}\varphi(x) &= \ell_0(x) + \ell_1(x) - \ell_2(x) + \ell_3(x)\\ &= 1 - 2\ell_2(x) && \text{(by (4.10))}\\ &= 1 + x(x-1)(x-3)\\ &= x^3 - 4x^2 + 3x + 1.\end{aligned}$$

On $[1,2]$, we have

$$\begin{aligned}\varphi(x) &= -\ell_0(x) + \ell_1(x) + \ell_2(x) - \ell_3(x)\\ &= 1 - 2\ell_0(x) - 2\ell_3(x) && \text{(by (4.10))}\\ &= 1 + \frac{1}{3}(x-1)(x-2)(x-3) - \frac{1}{3}x(x-1)(x-2)\\ &= 1 + \frac{1}{3}(x-1)(x-2)[(x-3) - x]\\ &= -1 + 3x - x^2.\end{aligned}$$

Now φ is symmetric with respect to $3/2$ and so it is sufficient to find the maximum of φ on $[0,1]$ and on $[1,2]$.
On $[0,1]$ the maximum occurs at $0, 1$ (where $\varphi = 1$) or at solutions of

$$0 = \varphi'(x) = 3x^2 - 8x + 3,$$

that is, at $x = (8 \pm \sqrt{28})/6 = \frac{1}{3}(4 \pm \sqrt{7})$. Of these, only $\frac{1}{3}(4 - \sqrt{7}) = 0.451416229$ lies in $[0,1]$ and since

$$\varphi(0.451416229) = 1.631130309,$$

we have

$$\max_{0 \le x \le 1} \varphi(x) = 1.631130309.$$

On $[1,2]$, $\max_{1 \le x \le 2} \varphi(x) = \varphi(3/2) = -1 + 4.5 - 2.25 = 1.25$.

Hence

$$\max_{0 \le x \le 3} \varphi(x) = 1.631.$$

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