The maximum error of p_1 is either $\frac{1}{2}$ or arises from solutions of

$$0 = f'(x) - p_1'(x) = 3x^2 - 2x - 1 = (3x+1)(x-1),$$

which are $-1/3$, 1. Now

$$\begin{aligned} f(-1/3) - p_1(-1/3) &= -\frac{1}{27} - \frac{1}{9} + \frac{1}{2} + \frac{1}{3} \\ &= \frac{1}{2} + \frac{5}{27}, \end{aligned}$$

which is greater than $\frac{1}{2}$ and has the same sign as $f(0) - p_1(0) = \frac{1}{2}$. Hence $-1/3$ replaces 0 in the reference.

Iteration 2 $\{-1, -1/3, 1\}$.
The equation to be solved are (5), (7) and

$$f(-1/3) - p(-1/3) = -\frac{1}{27} - \frac{1}{9} - a + \frac{1}{3}b = -h. \qquad (6')$$

Once again (5) and (7) give $b = 1$, so that

$$\left.\begin{array}{l} -a - 1 = h \\ -a + \frac{5}{27} = -h \end{array}\right\} \Rightarrow h = -\frac{16}{27}, a = -\frac{11}{27}.$$

Thus $p_2(x) = -\frac{11}{27} + x$ and $|h| = \frac{16}{27}$.

The maximum error of p_2 is either $\frac{16}{27}$ or arises from solutions of

$$0 = f'(x) - p_2'(x) = 3x^2 - 2x - 1 = (3x+1)(x-1),$$

which are $-1/3, 1$. Since $-1/3$ belongs to the reference, we deduce that p_2 is the b.m.a. p^* from $\mathcal{P}_1$ to f and so

$$\|f - p^*\|_\infty = \|f - p_2\|_\infty = 16/27.$$

Question 5

(a) Using Theorem 11.3, we have $\varphi_0(x) = 1$ and

$$\varphi_1(x) = x - \alpha_0,$$

where

$$\alpha_0 = \frac{(\varphi_0, x\varphi_0)}{\|\varphi_0\|^2} = \frac{\int_{-1}^1 x^3\,dx}{\int_{-1}^1 x^2\,dx} = \frac{0}{2/3} = 0$$

Thus $\varphi_1(x) = x$ and

$$\varphi_2(x) = (x - \alpha_1)x - \beta,$$

where

$$\alpha_1 = \frac{(\varphi_1, x\varphi_1)}{\|\varphi_1\|^2} = \frac{\int_{-1}^1 x^5\,dx}{\int_{-1}^1 x^4\,dx} = \frac{0}{2/5} = 0$$

and

$$\beta_1 = \frac{\|\varphi_1\|^2}{\|\varphi_0\|^2} = \frac{2/5}{2/3} = \frac{3}{5}.$$

Thus $\varphi_2(x) = x^2 - 3/5$.