Question 6

(a) The Fourier coefficients of f are

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta \, d\theta, \qquad n = 0, 1, 2, \ldots,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta \, d\theta, \qquad n = 1, 2, \ldots,.$$

Making the suggested change of variable $t = \theta + \pi/n$, and using the 2π–periodicity of the integrand, gives

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi+\pi/n}^{\pi+\pi/n} f(t-\pi/n)\cos n(t-\pi/n)dt \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t-\pi/n)[\cos nt\cos\pi + \sin nt\sin\pi]dt \\ &= -\frac{1}{\pi}\int_{-\pi}^{\pi} f(t-\pi/n)\cos nt\, dt. \end{aligned}$$

Hence

$$2a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(t) - f(t-\pi/n))\cos nt\, dt,$$

as required. Now let $\omega(\delta)$ be the modulus of continuity of f. Then

$$\begin{aligned} |2a_n| &\le \frac{1}{\pi}\int_{-\pi}^{\pi}\omega(\pi/n)|\cos nt|dt \\ &\le \frac{1}{\pi}\omega(\pi/n).1.2\pi \\ &= 2\omega(\pi/n). \end{aligned}$$

Since f is continuous, $\omega(\pi/n) \to 0$ as $n \to \infty$ and hence $|a_n| \to 0$ as $n \to \infty$.

A similar argument can be used with b_n since

$$\begin{aligned} \sin n(t-\pi/n) &= \sin nt\cos\pi - \cos nt\sin\pi \\ &= -\sin nt. \end{aligned}$$

Hence

$$2b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(f(t) - f(t-\pi/n))\sin\ nt\ dt,$$

and we again deduce that $|b_n| \le \omega(\pi/n) \to 0$ as $n \to \infty$.

(b) We shall apply Parseval's Theorem

$$\frac{1}{2}a_0^2 + \sum_{j=1}^{\infty}(a_j^2 + b_j^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\, dx$$

(which is obtained from Bessel's inequality by letting $n \to \infty$ and using Theorem 13.1) to $f(x) = |x|$. We have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\, dx = \frac{2}{\pi}\int_0^{\pi} x\, dx = \pi,$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi}|x|\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx\, dx \\ &= \frac{2}{\pi}\left[\frac{1}{n}x\sin nx\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{1}{n}\sin nx\, dx \\ &= -\frac{2}{n\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi} = \frac{2}{n^2\pi}((-1)^n - 1) \\ &= \begin{cases} -4/n^2\pi, & n \text{ odd}, \\ 0, & n \text{ even}, \end{cases} \end{aligned}$$

and

$$b_n = \frac{1}{n}\int_{-\pi}^{\pi}|x|\sin nx\, dx = 0,$$

since the integrand is odd.