(b) Let

$$L(f) = p(\alpha) - f(\alpha) = c_0 f(0) + c_1 f(1) + c_2 f'(\alpha) - f(\alpha)$$

so that L is a linear operator which depends on f, f' and annihilates $\mathcal{P}_2$ (by definition). Hence L annihilates $\mathcal{P}_1$ also and so, by the Peano Kernel Theorem,

$$L(f) = \int_0^1 K(\theta) f^{(2)}(\theta) d\theta, \qquad f \in C^{(2)}[0,1],$$

where

$$K(\theta) = \frac{1}{1!} L_x\{(x-\theta)_+\}.$$

(Note that we need $k = 1$ because $\|f^{(2)}\|_\infty$ appears on the right-hand side of the estimate.)

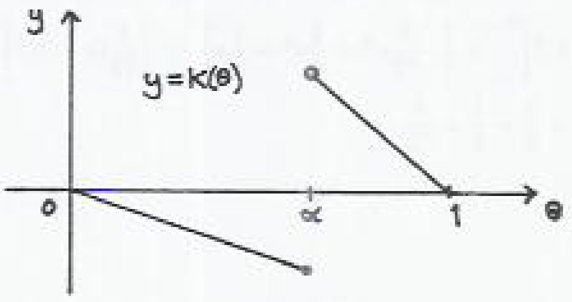
Now

$$K(\theta) = c_0(-\theta)_+ + c_1(1-\theta)_+ + c_2(\alpha-\theta)_+^0 - (\alpha-\theta)_+,$$

and, since K vanishes for $\theta \leq 0$ (because L annihilates $\mathcal{P}_1$),

$$K(\theta) = \begin{cases} c_0\theta, & 0 \leq \theta < \alpha, \\ c_1(1-\theta), & \alpha \leq \theta \leq 1, \end{cases}$$

(assuming that $(\alpha - \alpha)_+^0 = 1$, as defined in the Course Notes).



Hence

$$\begin{aligned} |p(\alpha) - f(\alpha)| &= |L(f)| \\ &\leq \left(\int_0^1 |K(\theta)| d\theta\right) \|f^{(2)}\|_\infty. \end{aligned}$$

Since

$$\begin{aligned} \int_0^1 |K(\theta)| d\theta &= |c_0| \int_0^\alpha \theta d\theta + |c_1| \int_\alpha^1 (1-\theta) d\theta \\ &= \frac{(\alpha-1)^2}{|2\alpha-1|} \int_0^\alpha \theta d\theta + \frac{\alpha^2}{|2\alpha-1|} \int_\alpha^1 (1-\theta) d\theta \\ &= \frac{(\alpha-1)^2\alpha^2}{2|2\alpha-1|} + \frac{\alpha^2(1-\alpha)^2}{2|2\alpha-1|} \\ &= \frac{\alpha^2(1-\alpha)^2}{|2\alpha-1|}, \end{aligned}$$

we obtain the required estimate.