Question 5

In this question we use the inner product

$$(f,g) = \int_{-1}^{1} x^2 f(x)g(x)\,dx, \quad f,g \in C[-1,1].$$

(a) Determine the monic polynomials $\phi_i \in \mathcal{P}_i, i = 0,1,2,$ which are orthogonal with respect to this inner product.

(b) Let $f(x) = |x|, \ -1 \le x \le 1$. Use your answer to part (a) to find the function p^* in $\mathcal{P}_2$ which minimises

$$\|f - p\|, \quad p \in \mathcal{P}_2,$$

for the norm arising from the above inner product.

(c) Use your answer to part (a) to obtain values of c_0, c_1, x_0, x_1 which make the approximation

$$\int_{-1}^{1} x^2 f(x)\,dx \approx c_0 f(x_0) + c_1 f(x_1)$$

exact when $f \in \mathcal{P}_3$.

Question 6

(a) Define the Fourier coefficients $a_0, a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ of a function f in $C_{2\pi}$. By making the change of variable $t = \theta + \pi/n$ in the definition of a_n, prove that

$$2a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(t) - f(t - \pi/n))\cos nt\,dt,$$

and deduce that $a_n \longrightarrow 0$ as $n \longrightarrow \infty$. Does the same argument apply to the sequence b_n?

(b) Consider the function f in $C_{2\pi}$ which satisfies $f(x) = |x|, \ -\pi \le x \le \pi$. Calculate the Fourier series of f and use this series to prove that

$$\sum_{j=1}^{\infty} \frac{1}{(2j-1)^4} = \frac{\pi^4}{96}.$$

Question 7

(a) Determine the cubic spline s with knots at $\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$ which interpolates $f(x) = |x|$ at the knots and which satisfies $s'(\pm 1) = f'(\pm 1)$.

(b) Determine the B-spline B of degree 2 with knots at $\{-1, 0, 1, 3\}$, giving your answer in piecewise polynomial form. Hence evaluate

$$\int_{-1}^{3} B(x)\,dx.$$

Question 8

For a given function f in $C^{(2)}[0,1]$, let p be the quadratic polynomial such that

$$p(0) = f(0), \quad p(1) = f(1) \text{ and } p'(\alpha) = f'(\alpha),$$

where $0 < \alpha < 1$, and $\alpha \ne \frac{1}{2}$.

(a) Express $p(\alpha)$ in terms of $f(0)$, $f(1)$, $f'(\alpha)$.

(b) Show that

$$|p(\alpha) - f(\alpha)| \le \frac{\alpha^2(1-\alpha)^2}{|1-2\alpha|}\|f^{(2)}\|_\infty.$$

[END OF QUESTION PAPER]