M343/D

The Open University

Third Level Course Examination 2000

Applications of Probability

Tuesday, 10 October, 2000 2.30 pm – 5.30 pm

Time allowed: 3 hours

This examination is in **TWO** parts. Part I carries 40% of the total available marks and Part II carries 60%.

You should attempt **FOUR** questions from Part I: each question carries 10 marks. You should attempt **THREE** questions from Part II: the questions in this part carry 20 marks each.

Write your answers to Parts I and II in **separate** answer books. Please start each question on a new page, and cross out rough working.

At the end of the examination

Fasten together your answer books for Parts I and II, using the paper fastener provided. Check that you have written your personal identifier and examination number on **each** answer book used. **Failure to do so will mean that your work cannot be identified.**

Important note

Note that if you have to solve an equation (or set of equations) in any question, then you should show all the intermediate steps in your solution, whether you solve the equation(s) algebraically or using an iterative procedure. If you simply state an answer without including such details, then *you will not be given any credit for the answer.*

PART I (Questions 1 to 6)

You should attempt **FOUR** *questions from this part of the examination, which carries 40% of the total available marks. Each question carries 10 marks.*

Question 1

The locations of breeding pairs of a certain species of bird in a wood are recorded and the number of breeding pairs in each of 20 equal-sized contiguous quadrats is counted. The data are as follows.

11	11	10	8	7
7	9	9	8	11
13	10	9	10	8
9	9	14	8	11

Commenting on the suitability of whatever test you adopt, use these data to investigate whether the breeding pairs could reasonably be supposed to be randomly distributed in the wood. If your test suggests that this is not the case, say what sort of pattern you think they might follow. Use a 5% significance level and state clearly the critical region of your test.

Question 2

In a Galton-Watson branching process starting with a single individual in generation zero, the offspring random variable X has a Poisson distribution with parameter 2.

(i) Calculate to four decimal places the probability that the process becomes extinct

(a) by the third generation,

(b) at the fourth generation.

(ii) Calculate to four decimal places the probability that the process eventually becomes extinct.

Question 3

In a study of weather patterns during the winter months in a Welsh seaside town, each day was classified as either sunny (state 1), cloudy but dry (state 2) or wet (state 3). The following matrix of transition probabilities was estimated from the data collected.

$$\mathbf{P} = \begin{matrix} \mathit{1} \\ \mathit{2} \\ \mathit{3} \end{matrix} \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.3 & 0.5 \\ 0.2 & 0.5 & 0.3 \end{pmatrix}$$

(i) If it is wet today, find the probability that it will be sunny tomorrow and cloudy but dry in two days' time.

(ii) According to the model, in the long run what proportion of winter days are sunny? What proportion are cloudy but dry? What proportion are wet?

(iii) Find the mean number of days from one sunny day to the next.

Question 4

Customers arrive at random at a ticket office where there is one assistant on duty at an average rate of three every twenty minutes. If the assistant is busy, an arriving customer joins the queue.

(i) What condition must the mean service time satisfy for the equilibrium queue size distribution to exist?

(ii) If the service time is constant and equal to four minutes, characterize the queue according to the standard notation. Find the mean queue size in this case.

(iii) Find the mean queue size if the service time is exponentially distributed with mean four minutes. In this case, for how long should a customer wishing to buy a ticket expect to have to queue (including the time spent being served)?

Question 5

The lifetime T (in months) of one of the components in a photocopier may be described by its hazard function.

$$h(t) = \frac{1}{10-t}, \quad 0 \leq t < 10.$$

(i) Find the survivor function $Q(t)$ of the component.

(ii) Say, giving reasons, whether the component is NBU, NWU or neither of these.

(iii) Find the mean lifetime of these components.

(iv) After several years, the photocopier is replaced; the old photocopier is sold. How many times should the new owner expect to have to replace the component during the next two years?

Question 6

An animal leaves its burrow near a river bank and moves up and down along the river bank foraging for food. Its distance upstream from its burrow after t minutes is denoted by $X(t)$, and may be reasonably modelled as an ordinary Brownian motion $\{X(t); t \geq 0\}$ with diffusion coefficient $\sigma^2 = 3$ (metres)^2 per minute.

(i) Find the probability that after 3 minutes the animal is less than 5 metres away from its burrow.

(ii) If the animal is observed to be 4 metres upstream from its burrow after 5 minutes, find the probability that it is downstream from its burrow after 15 minutes.

(iii) If the animal is observed to be 30 metres upstream from its burrow after an hour, find the probability that it was downstream from its burrow after 20 minutes.

PART II (Questions 7 to 12)

You should attempt **THREE** *questions from this part of the examination, which carries 60% of the total available marks. Each question carries 20 marks: the mark allocation for each part of each question is shown beside each part thus:* [4].

Question 7

(a) Give one example of a random phenomenon which could reasonably be modelled as a non-homogeneous Poisson process. Draw a rough sketch of the rate parameter $\lambda(t)$ for your example. [2]

(b) There are two methods of transport available from an airport terminal to a nearby hotel: taxi and minibus. Taxis arrive at the airport according to a Poisson process at an average rate of 12 per hour. Minibuses arrive according to a Poisson process at an average rate of 3 per hour.

(i) Find the probability that no taxi arrives during a ten-minute interval. [2]

(ii) What is the mean time between the arrival of transport at the terminal (taxi or minibus)? [2]

(iii) Calculate the probability that six vehicles arrive in half an hour, one of which is a minibus (the rest are taxis). [5]

(iv) Six vehicles arrive in half an hour one afternoon. What is the probability that exactly one of them is a minibus? [2]

(v) Use the table 'Random numbers from exponential distributions' given on page 43 of *Neave* to simulate (to the nearest minute) the times of arrival of vehicles between 10 am and 10.20 am one day. Use the twentieth row of the table (beginning 0.4676, 0.5834, ...). [4]

(vi) Describe a simulation method which uses single random digits for determining which of the vehicles in part (v) are minibuses. Illustrate your method using digits from the first row of the table on page 42 of *Neave*. [3]

Question 8

(a) Describe one example of a situation which could reasonably be modelled by a particle executing a simple random walk on the line with two absorbing barriers. Explain briefly why a simple random walk might be a suitable model and what the absorbing barriers represent. [3]

(b) A particle executes an unrestricted random walk on the line starting at the origin; its position after n steps is denoted by X_n. The ith step, Z_i, has the following distribution:.

$$P(Z_i = 1) = p, \quad P(Z_i = -3) = q = 1 - p.$$

(i) Derive the probability distribution of X_n; that is, find the probability $P(X_n = x)$, and state the range of X_n. Explain all the steps in your derivation. [5]

(ii) Find the following in terms of p and q.

(a) $E(X_{11})$

(b) $V(X_{11})$

(c) $P(X_{11} = 0)$

(d) $P(X_{11} = 3)$ [6]

(iii) Calculate in terms of p and q the probabilities u_0, u_4 and u_8 and hence find the probability that the particle returns to the origin for the first time after 8 steps. [6]

Question 9

(a) An immigration-birth process with arrival rate λ and birth rate β may be described by the probability statement

$$P(X(t+\delta t) = x+1 | X(t) = x) = (\lambda + \beta x)\delta t + o(\delta t),$$

where $X(t)$ denotes the size of the population at time t.

(i) Suppose that at a certain time there are x individuals in the population. State the probability distribution of the waiting time until the population size reaches $x+1$. [1]

(ii) Suppose that at time 0 the size of a population growing according to the above rule is 1. If $\lambda = 3\beta$, find in terms of β the mean of the waiting time until the population size reaches 4. [3]

(b) A partial differential equation for the probability generating function $\Pi(s,t)$ of the integer-valued random variable $X(t)$, which denotes the number of individuals alive at time t in an immigration-birth process $\{X(t); t \geq 0\}$, is

$$\frac{\partial \Pi}{\partial t} = -\lambda(1-s)\Pi - \beta s(1-s)\frac{\partial \Pi}{\partial s}.$$

(i) Rewrite this equation in Lagrange form and identify the functions f, g and h. [2]

(ii) Write down the auxiliary equations and show that their two solutions may be written

$$c_1 = \frac{s}{1-s}e^{-\beta t}, \quad c_2 = s^{\lambda/\beta}\Pi. \qquad [7]$$

(iii) Write down the general solution for the partial differential equation for $\Pi(s,t)$. [1]

(iv) Show that the particular solution for $\Pi(s,t)$, given that there are no individuals alive at time $t = 0$, is given by

$$\Pi(s,t) = \left(\frac{e^{-\beta t}}{1-(1-e^{-\beta t})s}\right)^{\lambda/\beta}. \qquad [4]$$

(v) If $\lambda = 3\beta$, find the probability that there is one individual alive at time t, given that there are no individuals alive at time 0. [2]

Question 10

(a) Briefly describe the main differences between the simple epidemic model and the general epidemic model for the spread of a disease through a community and between the way an epidemic develops in the two models. [4]

(b) In a family of total size 6, initially one person is suffering from a disease which is incurable, but not serious; the other five are susceptible. Infectives are not isolated and the infective contact rate is $\beta = 0.2$ per day.

Find the mean and standard deviation of the waiting time until the whole family has the disease. [7]

(c) In another family of total size 6, initially two people are suffering from a certain disease; the other four members of the family are susceptible. An individual who catches this disease remains infectious for a time which is exponentially distributed with mean 4 days. The infective contact rate is $\beta = \frac{5}{8}$ per day.

(i) Show that the epidemic parameter ρ is equal to 2. [1]

(ii) Use a stochastic model to calculate the probability that more than one of the four susceptible members of the family catches the disease before the epidemic dies out. [8]

Question 11

(a) Briefly describe how a population which is stable but not stationary might evolve. [3]

(b) The life table function for members of a particular stationary bird population is

$$Q(x) = \left(1 - \frac{x}{6}\right)^2, \quad 0 \le x < 6,$$

where x is measured in years.

(i) Find the median age at which birds die. [2]

(ii) Find the mean lifetime of birds in this population. What proportion of birds live more than twice the mean lifetime? [3]

(iii) Find the median age of the birds in the population. [4]

(iv) (a) What proportion of birds live for less than one year?

(b) At any time, what proportion of birds are less than one year old? [3]

(v) (a) Draw a rough sketch of the p.d.f. of the age-distribution of the population.

(b) Now suppose that the population with the life table function $Q(x)$ given above is stable but not stationary. Describe briefly how the age-distribution for the population would differ from that of the stationary population if the population is (i) growing, (ii) declining. Illustrate your answer in each case by drawing a rough sketch of the p.d.f. of the age-distribution.

Note: *No calculations are required for this part of the question.* [5]

Question 12

(i) In a large population of insects which are introduced into a habitat, wing colour is controlled by a single gene: homozygous insects **BB** have dark-brown wings and heterozygous insects **Bb** have light-brown wings; homozygous insects **bb** have orange wings. Initially 30% of the insects have dark-brown wings, 20% have light-brown wings and 50% have orange wings.

The insects reproduce in discrete generations by random mating. Describe what will happen to the proportions of insects of each colour in future generations, mentioning any theorem that you use. [7]

(ii) The wing-type of this population of insects is also controlled by a single gene; this gene is inherited independently of the gene controlling wing colour. Doubly dominant insects **AA** and heterozygotes **Aa** have plain wings; homozygous recessives **aa** have spotted wings. After many generations it is observed that only 9% of the insects have spotted wings.

(a) What proportion of the population is 'pure' plain-winged (doubly-dominant **AA**)? [3]

(b) What proportion of plain-winged insects is **AA**? [3]

(c) If two randomly-selected plain-winged insects are mated, what proportion of their offspring do you expect to have spotted wings? [5]

(iii) After many generations, what proportion of insects in the population will have plain orange wings? [2]

[END OF QUESTION PAPER]