Question 11

Let K be the subfield of $\mathbf{C}$ which is the splitting field of the irreducible polynomial $t^5 - 3$ in $\mathbf{Q}[t]$, and let G be the Galois group $\Gamma(K : \mathbf{Q})$.

(i) Find $\alpha, \beta \in \mathbf{C}$ such that $K = \mathbf{Q}(\alpha, \beta)$. [4]

(ii) Find the degree $[K : \mathbf{Q}]$. [2]

(iii) Show that the group G has a normal subgroup N such that both N and G/N are cyclic; identify the groups N and G/N. [4]

Question 12 (Unit 13)

Determine whether or not the following ruler and compass constructions are possible, giving a justification in each case.

(i) Construction of a regular 1995-gon. [3]

(ii) Construction of an angle of $4\pi/105$. [4]

(iii) Construction of a rectangle, three times as long as it is wide, equal in area to a given circle. [3]

Question 13 (Unit 14)

Show that the polynomial

$$t^5 - 6t^3 + 2$$

in $\mathbf{Q}[t]$ is not soluble by radicals over $\mathbf{Q}$, giving clear references to any theorems used. [10]

Question 14 (Unit 15)

Let p be a prime, let f be an irreducible polynomial of degree 15 in $\mathbf{Z}_3[t]$ and let L be a splitting field for f over $\mathbf{Z}_3$.

(i) Prove that every irreducible polynomial of degree 15 in $\mathbf{Z}_3[t]$ splits over L. [4]

(ii) Let g be an irreducible polynomial of degree 5 in $\mathbf{Z}_3[t]$. Prove that g splits over L, justifying your answer. [3]

(iii) Find a polynomial of degree 2 in $\mathbf{Z}_3[t]$ which does not split over L, justifying your answer. [3]

Question 15 (Unit 16)

Consider the following statements about ordered fields. For each one say whether the statement is true or false, justifying your answer with a brief proof or counter-example.

(i) If K is an ordered field then every subfield M of K is an ordered field. [2]

(ii) If K is an ordered field, $x \in K$ and $x \geq 0$, then $-x \leq 0$. [2]

(iii) If K is an ordered field and $k \in K$ then $k + k + k$ is not zero unless $k = 0$. [2]

(iv) Every ordered field has characteristic zero. [2]

(v) Every ordered field has a proper algebraic extension. [2]

[END OF QUESTION PAPER]