Question 5

(i) Suppose that G is a simple group whose centre, $Z(G)$, is not the trivial subgroup. Prove that G is cyclic of prime order. [6]

(ii) Give an example of a group which is not soluble and which has a non-trivial centre, justifying your choice. [4]

Question 6

Let G be a group of order 168 which has exactly 6 conjugacy classes. Three of these conjugacy classes have sizes 24, 24 and 42.

(i) Find the sizes of the other three conjugacy classes. [3]

(ii) Show that G is a simple group. [7]

Question 7

For each of the following field extensions state, with brief reasons, whether or not it is algebraic. For each of the extensions which is algebraic, state, with brief reasons, whether or not it is normal.

(i) $\mathbf{Q}(4-\sqrt[3]{5}):\mathbf{Q}$ [3]

(ii) $\mathbf{Q}(\pi^2):\mathbf{Q}(\pi^4)$ [4]

(iii) $\mathbf{Q}(i,\sqrt{2}):\mathbf{Q}$, where $i^2=-1$ [3]

Question 8

Let K and L be fields with $K\subseteq L$. Let $\alpha\in L$ and suppose that α is algebraic over K.

(i) Prove that $K(\alpha^3)\subseteq K(\alpha)$. [1]

(ii) Prove that α^3 is algebraic over K. [3]

(iii) Prove that $[K(\alpha):K(\alpha^3)]=1$, 2, or 3. [3]

(iv) Give examples with $K=\mathbf{Q}$ and $\alpha\notin\mathbf{Q}$ to show that all of the cases in part (iii) can occur. [3]

Question 9

Let K be the field $\mathbf{Q}(\sqrt[3]{4},\omega)$, where ω is a non-real cube root of 1. You may assume that K is a splitting field for t^3-4 over $\mathbf{Q}$.

(i) Find the Galois group $\Gamma(K:\mathbf{Q})$, specifying, for each element of the group, its effect on $\sqrt[3]{4}$ and on ω. [5]

(ii) Find all subfields of K which are normal extensions of $\mathbf{Q}$. [5]

Question 10

Let m be a polynomial of degree 3 in $\mathbf{Q}[t]$, let K be the splitting field for m over $\mathbf{Q}$ and let G be the Galois group of $K:\mathbf{Q}$.

(i) What are the possible orders of G? For each possible order, identify the possible groups G. [7]

(ii) For each of the following polynomials m, identify the corresponding Galois group, G

(a) $m(t)=(t-1)^3$,

(b) $m(t)=t^3-1$,

(c) $m(t)=t^3-2$. [3]