Question 6

For *each* of the following four properties, describe (with brief justification) a group G of order 120 possessing that property.

(i) G is abelian but not cyclic. [2]

(ii) G is metabelian but not abelian. [3]

(iii) G is soluble but not metabelian. [3]

(iv) G is not soluble. [2]

Question 7

(i) For each of the following field extensions state, with brief reasons, whether or not it is algebraic.

(a) $\mathbf{Q}\left(\cos\frac{\pi}{4}\right) : \mathbf{Q}$ [2]

(b) $\mathbf{Q}(\pi^2) : \mathbf{Q}$ [2]

(ii) For each of the following finite field extensions state, with brief reasons, whether or not it is normal.

(a) $K : \mathbf{Z}_7$, where $K = \mathbf{Z}_7[t]/\langle t^2 + t + 4\rangle$ [2]

(b) $\mathbf{Q}(\sqrt[3]{3}, \sqrt{7}) : \mathbf{Q}$ [2]

(c) $\mathbf{Q}(\sqrt[3]{2}, \sqrt{7}, \omega) : \mathbf{Q}$ (where $\omega \in \mathbf{C}, \omega^3 = 1, \omega \neq 1$) [2]

Question 8

Let K and L be fields with $K \subseteq L$. Let $\alpha \in L$ and suppose that α is algebraic over K.

(i) Prove that $K(\alpha^3) \subseteq K(\alpha)$. [1]

(ii) Prove that α^3 is algebraic over K. [3]

(iii) Prove that $[K(\alpha) : K(\alpha^3)] = 1$, 2, or 3. [3]

(iv) Give examples with $K = \mathbf{Q}$ and $\alpha \notin \mathbf{Q}$ to show that all of the cases in part (iii) can occur. [3]

Question 9

Let K be the subfield $\mathbf{Q}(\sqrt{3}, i\sqrt{2})$ of $\mathbf{C}$ and let G be the group $\Gamma(K : \mathbf{Q})$.

(i) Find G by specifying for each element of G its effect on $\sqrt{3}$ and $i\sqrt{2}$. Is G cyclic? Justify your answer. [6]

(ii) Find all the subfields of K, specifying each subfield in the form $\mathbf{Q}(a, b, c, \ldots)$, for appropriate elements $a, b, c, \ldots$. [4]

Question 10

Let f be an irreducible cubic polynomial in $\mathbf{Q}[t]$, let Σ be a splitting field for f over $\mathbf{Q}$, and let G be the Galois group $\Gamma(\Sigma : \mathbf{Q})$.

(i) Prove that G is isomorphic to a subgroup of S_3. [4]

(ii) Prove that
either (a) $[\Sigma : \mathbf{Q}] = 3$,
or (b) there is exactly one field M such that $\mathbf{Q} \subset M \subset \Sigma$ and $M : \mathbf{Q}$ is normal.

In case (b), state the degree of $M : \mathbf{Q}$. [6]