This paper is not to be removed from the Examination Halls

UNIVERSITY OF LONDON **996 D021 ZA**

BSc degrees and Diplomas for Graduates in Economics, Management, Finance and the Social Sciences, the Diploma in Economics and Access Route for Students in the External Programme

Mathematics

Thursday, 11 May 2006 : 10.00am to 1.00pm

Candidates should answer **EIGHT** of the following **ELEVEN** questions: **SIX** from Section A (10 marks each) and **TWO** from Section B (20 marks each). **Candidates are strongly advised to divide their time accordingly.**

Graph paper is provided. If used, it must be securely fastened inside the answer book.

Calculators may **not** be used for this paper.

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SECTION A (60% of the marks)

Please answer all **six** questions in this section (10 marks each).

1. Determine the following integrals.

$$\int \frac{3x^3 + 2x^2 + 4x + 4}{x^2 + 1}\, dx$$

$$\int \frac{\ln x}{x^2}\, dx$$

$$\int \frac{\cos(\ln x)}{x}\, dx$$

$$\int \frac{\cos x}{(\sin x)^2 + 2\sin x + 2}\, dx$$

2. Find the eigenvalues of the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 5 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

and obtain one eigenvector for each eigenvalue.

3. (a) Solve by separating the variables

$$\frac{dy}{dx} = (y-1)(y-2)\sin x \cos x.$$

(b) Find a particular solution of the form $y = e^x(A\cos x + B\sin x)$ of the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 10e^x \sin x.$$

Hence find the general solution.

4. Given that

$$z = \frac{xy+1}{xy-1}$$

find the partial derivatives

$$\frac{\partial z}{\partial x}, \ \frac{\partial z}{\partial y}.$$

Suppose that y is fixed at 1, but that x increases by 1% from $x = 2$. Using the partial derivatives you have obtained, find an approximation to the percentage change in z.

5. For $t = 0, 1, 2, \ldots$, q_t and p_t satisfy the equations

$$\begin{aligned} p_t &= q_t \\ q_t &= 4 - p_t \\ q_t &= 1 + 0.5p_{t-1}. \end{aligned}$$

Find expressions for p_t and q_t. How do p_t and q_t behave as t tends to infinity?

6. In the 'Verhulst model' the population level N grows according to the law

$$\frac{dN}{dt} = cN(S-N),$$

where c is a positive constant and S represents a 'saturation level' for the population. Thus the population growth depends directly on the current population level and on its divergence from the saturation level.

Use the method of partial fractions to solve the differential equation and so find a formula for $N(t)$ in terms of the initial population level $N(0)$ and the constants c, S. Show that as $t \to \infty$

$$N(t) \to S.$$

Suppose that $0 < N(0) < S/2$. Deduce that the population level is always increasing and show that the population level initially is convex, but that after the population level has passed the half-saturation level, i.e. $N > S/2$, it is concave and ultimately almost flat.

If $N(0) > S$ what behaviour does the model predict (i.e determine whether the population is increasing or decreasing with time)?

SECTION B (40% of the marks)

Please answer **two** questions from this section (20 marks each).

7. (a) Show that the function

$$2x^3 - 6x^2y - 3x^2 + 12y^3 + 12y^2 + 3y$$

has two saddle-points and one local minimum.

(b) A company has the following production function

$$q(k,l) = \left(k^{1/4} + l^{1/4}\right)^4.$$

It has to produce an amount Q when the price of capital is $v = 8$ and the price of labour is $w = 1$, so the total cost when k units of capital and l units of labour are employed is $8k + l$. What is the optimal choice of capital and labour to produce Q at minimum cost.

8. (a) Use an integrating factor to solve the equation.

$$\frac{dy}{dx} + y \sin x = \sin 2x.$$

(b) Check that the following equation is exact and hence solve it.

$$(y\cos(xy) + y\ln y + 2xy) + (x\cos(xy) + x + x\ln y + x^2)\frac{dy}{dx} = 0.$$

(c) Show that the right-hand side of the following differential equation is homogeneous and hence solve the equation.

$$\frac{dy}{dx} = \frac{2y^2 + xy - 6x^2}{xy}.$$

9. (a) Express the following system of simultaneous equations in matrix form, and solve it using Cramer's rule or any *matrix method.*

$$x + 2y + 4z = 7,$$

$$x + 3y + 9z = 16$$

$$x + 4y + 16z = 29.$$

(b) The quantity of a commodity supplied to the market when the selling price is P is believed to take the form:

$$Q = a + bP + cP^2$$

for some constants a, b, c. It is known that when $P = 2$ the quantity supplied is $Q = 7$; when $P = 3$ the quantity supplied is $Q = 16$ and when $P = 4$ the quantity supplied is $Q = 29$. Find a system of three linear equations in the unknowns a, b, c. Solve this to determine the constants a, b, c. If the formula is to be believed, what would be the quantity supplied if the price was $P = 1$?

10. (a) Sketch the curve

$$\frac{x^2 + 2x + 3}{x + 3}$$

taking care to obtain the stationary points and the oblique asymptote accurately.

(b) The positive quantity y is related to x by the equation

$$x^4y^3 + 4x^2y^2 - 2x^5y = 3.$$

Find dy/dx when $x = 1$.

[Hint: it may help to note that $y^3 + 4y^2 - 2y - 3 = (y - 1)(y^2 + 5y + 3)$.]

(c) Find $\frac{\partial q}{\partial k}$ and $\frac{\partial q}{\partial l}$ when q, k, l are related by the equation:

$$q^3k + 2l^2k + q^2l^3 = 0$$

11.(a) Show that $y = Ax^\alpha + Bx^\beta$ is the general solution of the equation

$$x^2 y''(x) - mxy'(x) - ny = 0$$

provided α, β are the distinct roots of

$$\gamma^2 - (1+m)\gamma - n = 0.$$

Solve this quadratic equation when $m = 1, n = 8$ and when $m = 1, n = 3$.

(b) When the current investments of an investment bank are valued at x, the value of the bank to its depositors is $D(x)$ and is modelled as follows.

(i) If $x = b$, the bank goes out of business, and $D(x) = b$.

(ii) If $b \leq x \leq k$ then the regulating authority may close the bank and distribute the value x to the depositors, but only if it inspects the bank. In this range the value $D(x)$ is modelled by the solution to the differential equation:

$$x^2 D'' + (\rho - \delta)xD' - (\rho + \lambda)D + C + \lambda x = 0.$$

(iii) If $x \geq k$ the value $D(x)$ is modelled by the solution to the differential equation:

$$x^2 D'' + (\rho - \delta)xD' - \rho D + C = 0.$$

Here, in appropriate units, C is the payout rate to its depositors whilst the bank is in business, ρ measures the interest rate, δ is the rate at which the bank pays dividends to its mangement, and λ measures the regulator's inspection intensity.

Assume $\rho = 3,\ \delta = 2,\ \lambda = 5$ and $C = 4$.

Show that the equation in (ii) has a particular solution of the form $Kx + L$ and hence find the general solution to the equation by using (a).

Show that the equation in (iii) has a particular solution C/ρ.

Using part (a) again, find the most general solution to this equation such that

$$\lim_{x \to \infty} D(x) = C/\rho.$$

Assuming that $b = 0.5, k = 1$, write down conditions on the solutions to (ii) and (iii), so that they have equal value and equal slopes at $x = k = 1$.

END OF PAPER