- 1. (a) Prove the following identities, either by algebraic manipulation or by combinatorial argument:
 - (i) $\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c};$
 - (ii) $\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}^2$.
- (b) How many ways are there to choose 13 cards from a standard deck such that at least one of the following is true: exactly 3 of them are diamonds, exactly 4 of them are hearts, exactly 5 of them are spades? (In this part and the next, you may express your answers in terms of binomial coefficients or factorials without expanding them.)
- (c) In how many ways can the letters of the word "undefended" be rearranged? Of these, how many do not include the sequence NN? How many do not include the sequence EE?

[20 marks]

- **2.** (a) Let a, b, c be positive integers such that 2b > a + c. Let m_1, \ldots, m_b be a sequence of positive integers whose sum is a. Prove that there is a subsequence consisting of consecutive terms $m_r, m_{r+1}, \ldots, m_s$ whose sum is c.
- (b) Suppose given five points P_1, \ldots, P_5 in a square (including boundary) of side length 1. Prove that there are P_i, P_j with $i \neq j$ such that the distance from P_i to P_j is at most $1/\sqrt{2}$. Show how to choose P_1, \ldots, P_5 so that the minimum distance is exactly $1/\sqrt{2}$.
- (c) Let $S = \{0, 1, ..., 22\}$ and let T be a subset of S with 7 elements. Prove that there are two distinct subsets of T with the same sum. [20 marks]

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- **3.** (a) State Hall's theorem. State a version of Hall's theorem that applies to matchings of two sets S and T in which each element of S must be paired with several elements of T.
- (b) Eight people A, B, C, D, E, F, G, H are to do eight tasks 1, 2, ..., 8. Only some of the people can do each task, as shown in the following table:

Α	1, 3, 5	Е	1, 3, 6
В	1, 4, 7, 8	F	3, 4, 8
С	2, 7, 8	G	5, 6
D	1, 5, 6	Н	1, 5

- (i) Show that seven of the tasks can be assigned, but not all eight.
- (ii) Suppose it is decided that B will perform two of the tasks. Prove that it is now possible to assign all of the tasks. (One of the people will be idle.)
- (iii) In the situation of (ii), is it possible to arrange for G not to be assigned a task?

[20 marks]

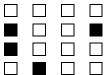
4. Four medical students x_1, \ldots, x_4 have applied for four hospital residencies y_1, \ldots, y_4 . The students and the hospitals have the following preference orders:

- (i) State the definition of a stable matching. Find (a) the stable matching that is optimal from the students' point of view, (b) the stable matching that is optimal from the hospitals' point of view.
- (ii) Determine whether there is a stable matching in which x_4 is paired with (a) y_1 ; (b) y_4 . [20 marks]

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5. (i) Define a rook polynomial. State a formula that relates the rook polynomial of a board to that of its complement.

(ii) Hence,	or	otherwise,	determine	the	rook	polynor	$_{ m nial}$	of the	4 ×	: 4	board
shown h	ere.											



(where \square represents a square that belongs to the board and \blacksquare one that does not).

(iii) Five people A, B, C, D, E are to be assigned to five offices 1, 2, 3, 4, 5, subject to the following constraints: A and B are unwilling to accept office 1, B refuses office 2, C and E do not want office 3, D rejects office 4, and everyone is happy to accept office 5. In how many ways can these people be assigned offices? [20 marks]

6. (a) Solve each of the following recurrences or systems of recurrences. In each case give explicit formulas both for the sequence a_i and the generating function $\sum a_i t^i$.

(i)
$$a_0 = 2, a_1 = 2, a_i = 4a_{i-1} - 4a_{i-2}$$
 for $i > 1$;

(ii) $a_0 = 1, b_0 = -1, a_i = 2a_{i-1} + b_{i-1}$ for $i > 0, b_i = a_{i-1} + 2b_{i-1}$ for i > 0. (Solve only for the a_i , not for the b_i .)

(iii)
$$a_0 = 1, a_i = 3a_{i-1} + 2^i \text{ for } i > 0.$$

(b) For all positive integers i, let M_i be an $n \times n$ matrix with 2 on the diagonal, 1 immediately above it, and -1 immediately below it. For example,

$$M_5 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

Let $a_i = \det M_i$ and $a_0 = 1$. Give an explicit formula for a_i and for $\sum a_i t^i$.

[20 marks]

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7. Give generating functions for (a) the number of partitions of n with no even part repeated, (b) the number of partitions of n with no part a multiple of 4, (c) the number of partitions of n with no part repeated more than three times, and show that they are all equal.

[20 marks]

8. (i) Let a, b, c be complex numbers such that

$$a + b + c = 3$$

 $a^{2} + b^{2} + c^{2} = 5$
 $a^{3} + b^{3} + c^{3} = 7$.

Determine the cubic polynomial whose roots are a, b, c. Find $a^4 + b^4 + c^4$ and $a^5 + b^5 + c^5$ (preferably by using Newton's identities).

(ii) With a, b, c as above, determine the cubic polynomial whose roots are a + b, b + c, c + a. [20 marks]