THE UNIVERSITY
of LIVERPOOL

SOLUTIONS FOR MATH744 (MAY 2006)

SECTION A

1.

(a) $\{v_1, \dots, v_k\}$ are linearly independent if the only solution to $\lambda_1 v_1 + \dots \lambda_k v_k = 0$ is given by $\lambda_1 = \dots = \lambda_k = 0$. (Alternatively: none of $v_1, \dots, v_k$ can be written as a linear combination of the other vectors.)

[1 mark]. *Standard definition from lectures.*

(b) *First method*: First put u_1, u_2, u_3 as the rows of a matrix, and use row operations to reduce to echelon form. Solution:

$$\begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \\ 3 & 0 & 1 \end{pmatrix} \longrightarrow \dots \longrightarrow \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $(3,0,1), (0,3,-2)$ is a basis of U, and the dimension is 2.

Second method: Find a nontrivial solution to the equation $\lambda u_1 + \mu u_2 + \nu u_3 = 0$; e.g. $2(1,-1,1) + (1,2,-1) - (3,0,1) = (0,0,0)$. So the three vectors are linearly dependent, so $\dim U < 3$. On the other hand, there are clearly two linearly independent vectors among the three vectors given (any pair will do), so $\dim U \geq 2$.

Remark: An easy way to check whether a given basis for U is correct is to note that $U = \{(x,y,z) : x - 2y = 3z\}$.

[3 marks]. *Standard exercise.*

(c) *First method*: Again, put w_1, w_2, w_3 as the rows of a matrix, and use row operations to reduce to echelon form:

$$\begin{pmatrix} -4 & 1 & -2 \\ 2 & 1 & 0 \\ 5 & 1 & 1 \end{pmatrix} \longrightarrow \dots \longrightarrow \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore the space W also has the basis $\{(3,0,1), (0,3,-2)\}$, and so $U = W$.

Second method: Since we have already computed the dimension of U as 2, and the dimension of W is clearly at least 2, it is enough to check that $W \subset U$; i.e., each of the vectors w_j belongs to U. This can be done, for example, by writing them as linear combinations of u_1 and u_2 (again solving a system of linear equations):

$$w_1 = -3u_1 - u_2, \quad w_2 = u_1 + u_2, \quad w_3 = 3u_1 + 2u_2.$$

[3 marks]. *Standard exercise.*

(d) Now $V = \mathrm{Pol}_3(\mathbb{R})$ is the vector space of polynomials with real coefficients of degree at most three, and

$$U := \{(2a+b)x^3 + ax^2 - (2a+b)x - b : a, b \in \mathbb{R}\} \quad \text{and}$$
$$W := \{(b-a)x^3 + cx^2 + (a-b)x - c : a, b, c \in \mathbb{R}\}.$$

Let $v_1 = (2a_1+b_1)x^3 + a_1x^2 - (2a_1+b_1)x - b_1$ and $v_2 = (2a_2+b_2)x^3 + a_2x^2 - (2a_2+b_2)x - b)2$ be arbitrary elements of U, and let $\mu, \lambda \in \mathbb{R}$. Setting $a := \lambda a_1 + \mu a_2$ and $b := \lambda b_1 + \mu b_2$, a simple calculation shows

$$\lambda v_1 + \lambda v_2 = (2a+b)x^3 + ax^2 - (2a+b)x - b \in U.$$

So U is a subspace of V.

[3 marks]. *Seen similar in exercises*

(e) By definition of U, $(2x^3 + x^2 - 2x, x^3 - x - 1)$ is a spanning set of U. Since the two vectors are clearly linearly independent, it is also a basis. Thus the dimension of U is two. Similarly, $(x^3 - x, x^2 - 1)$ is a basis for W, and the dimension of W is also two.

(We mention here that $U = \{ax^3 + bx^2 + cx + d : a + c = 0 \text{ and } a = 2b - d\}$ and $W = \{ax^3 + bx^2 + cx + d : a + c = 0 \text{ and } b + d = 0\}$; for either space, any two linearly independent vectors would provide an acceptable answer.)

[4 marks]. *Seen similar in exercises*

To find $U \cap W$, we need to decide when an arbitrary vector v of V belongs to both U and W. There are several ways of doing this:

(i) Using the definition of U and W, we need to solve the equations

$$\begin{aligned} 2a_1 + b_1 &= b_2 - a_2 \\ a_1 &= c_2 \\ -2a_1 - b_1 &= a_2 - b_2 \\ -b_1 &= -c_2. \end{aligned}$$

So we have $U \cap W = \{3ax^3 + ax^2 - 3ax - a : a \in \mathbb{R}\}$. Thus $3x^3 + x^2 - 3x - 1$ is a basis for $U \cap W$, and $\dim(U \cap W) = 1$.

(ii) Similarly, we can use the bases for U and W and solve the equation

$$\lambda_1(2x^3 + x^2 - 2x) + \mu_1(x^3 - x - 1) = \lambda_2(x^3 - x) + \mu_2(x^2 - 1).$$

Solving this equation, we get $\mu_1 = \mu_2 = \lambda_1$, and $\lambda_2 = 2\lambda_1 + \mu_1 = 3\lambda_1$. Again, we obtain $3x^3 + x^2 - 3x - 1$ as a basis for $U \cap W$.

(iii) It is also sufficient to exhibit one single vector which belongs to both U and W; for example, the vector $3x^3 + x^2 - 3x - 1$ (which corresponds to $a = b = 1$ in the definition of U, and to $a = 0, b = 3, c = 1$ in the definition of W). Since $U \neq W$, the dimension of $U \cap W$ must then be 1.

[4 marks]. *Seen similar in exercises*

We thus have $\dim(U + W) = \dim U + \dim W - \dim(U \cap W) = 3$. (Note that we have $U + W = \{ax^3 + bx^2 + cx + d : a + c = 0\}$.) Since $U \cap W \neq \{0\}$, $U + W$ is not the direct sum of U and W.

[2 marks]. *Standard exercise.*

20 marks in total for Question 1

2.

(a) We compute:

$$\begin{aligned} f(u_1,u_1) &= 2\cdot 1\cdot 1 + 1\cdot(-1) + 2\cdot(-1)\cdot(-1) = 3,\\ f(u_1,u_2) &= 2\cdot 1\cdot 1 + 1\cdot(-2) + 2\cdot(-1)\cdot(-2) = 4,\\ f(u_2,u_1) &= 2\cdot 1\cdot 1 + 1\cdot(-1) + 2\cdot(-2)\cdot(-1) = 5.\\ f(u_2,u_2) &= 2\cdot 1\cdot 1 + 1\cdot(-2) + 2\cdot(-2)\cdot(-2) = 8. \end{aligned}$$

So, the matrix of f wrt u_1, u_2 is

$$A = \begin{pmatrix} 3 & 4 \\ 5 & 8 \end{pmatrix}.$$

[**2 marks**] *Standard exercise.*

Similarly,

$$\begin{aligned} f(v_1,v_1) &= 2\cdot(-2)\cdot(-2) + (-2)\cdot 1 + 2\cdot 1\cdot 1 = 8,\\ f(v_1,v_2) &= 2\cdot(-2)\cdot 5 + (-2)\cdot 1 + 2\cdot 1\cdot 1 = -20,\\ f(v_2,v_1) &= 2\cdot 5\cdot(-2) + 5\cdot 1 + 2\cdot 1\cdot 1 = -13,\\ f(v_2,v_2) &= 2\cdot 5\cdot 5 + 5\cdot 1 + 2\cdot 1\cdot 1 = 57, \end{aligned}$$

So, the matrix of f wrt v_1, v_2 is

$$B = \begin{pmatrix} 8 & -20 \\ -13 & 57 \end{pmatrix}.$$

[**2 marks**] *Standard exercise.*

To compute the change-of-basis matrix, we write v_j as linear combinations of the u_j. (Again, this will involve solving a system of linear equations.)

$$\begin{aligned} (-2,1) &= -3\cdot(1,-1) + 1\cdot(1,-2)\\ (5,1) &= 11\cdot(1,-1) - 6\cdot(1,-2). \end{aligned}$$

So the change-of-basis matrix is

$$P = \begin{pmatrix} -3 & 11 \\ 1 & -6 \end{pmatrix}.$$

Alternatively, we can obtain P as the composition of change-of-basis matrices from the given bases to the standard basis:

$$P = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -2 & 5 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} -2 & 5 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 11 \\ 1 & -6 \end{pmatrix}.$$

Finally, it is easily checked that

$$P^T A P = \begin{pmatrix} -3 & 1 \\ 11 & -6 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 5 & 8 \end{pmatrix} \begin{pmatrix} -3 & 11 \\ 1 & -6 \end{pmatrix} = B.$$

[**3 marks**]. *Seen similar in exercises.*

(b) The matrix of the quadratic form

$$q(x, y, z) = 4x^2 - 4y^2 + z^2 + 6xy.$$

with respect to the standard bases is

$$A = \begin{pmatrix} 4 & 3 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

[2 marks]. *Standard exercise.*

We can find a basis with respect to which q is diagonal by finding a basis consisting of orthogonal eigenvectors of A. The characteristic polynomial is

$$\begin{aligned} \det(\lambda I - A) &= \left| \begin{pmatrix} (\lambda - 4) & -3 & 0 \\ -3 & (\lambda + 4) & 0 \\ 0 & 0 & (\lambda - 1) \end{pmatrix} \right| \\ &= (\lambda - 1) \left| \begin{pmatrix} (\lambda - 4) & -3 \\ -3 & (\lambda + 4) \end{pmatrix} \right| \\ &= (\lambda - 1)(\lambda^2 - 16 - 9) \\ &= (\lambda - 1)(\lambda - 5)(\lambda + 5), \end{aligned}$$

so the eigenvalues are 5, -5 and 1. Solving the corresponding linear equations gives eigenvectors $(3, 1, 0)$, $(1, -3, 0)$ and $(0, 0, 1)$. The desired matrix P is thus given by

$$P = \begin{pmatrix} 3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The desired diagonal matrix is

$$D = P^T A P = \begin{pmatrix} 50 & 0 & 0 \\ 0 & -50 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

[8 marks]. *Seen somewhat similar in exercises.*

The diagonal matrix has full rank, so the rank of q is 3. The signature is the number of positive entries minus the number of negative entries, and is thus 1. The surface is a hyperboloid of one sheet.

[3 marks]. *Standard exercise.*

20 marks in total for Question 2

3.

(a) Let $e_1 = x^2, e_2 = x, e_3 = 1$. Then

$$\varphi(e_1) = \varphi(x^2) = 3x^2 - 2x + 2 = 3 \cdot e_1 - 2 \cdot e_2 + 2 \cdot e_3,$$

so that the first column of the matrix should have entries $3, -2, 2$. Proceeding similarly for e_2 and e_3, we get

$$M = \begin{pmatrix} 3 & 1 & 0 \\ -2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

[3 marks] *Seen similar in exercises.*

(b) We now compute

$$\begin{aligned} \det(\lambda I - M) &= \begin{vmatrix} (\lambda-3) & -1 & 0 \\ 2 & (\lambda-1) & -1 \\ -3 & -1 & (\lambda-1) \end{vmatrix} \\ &= (\lambda-1)\begin{vmatrix} (\lambda-3) & -1 \\ 2 & (\lambda-1) \end{vmatrix} + \begin{vmatrix} (\lambda-3) & -1 \\ -2 & -1 \end{vmatrix} \\ &= (\lambda-1)((\lambda-3)(\lambda-1)+2) + (1-\lambda) \\ &= (\lambda-1)(\lambda^2 - 4\lambda + 4) = (\lambda-1)(\lambda-2)^2. \end{aligned}$$

So the eigenvalues of λ are 1 and 2.

[4 marks] *Standard exercise.*

(c) To find the eigenvectors corresponding to these eigenvalues, we must solve the equations $(M - I)v = 0$ and $(M - 2I)v = 0$:

$$\begin{pmatrix} 2 & 1 & 0 \\ -2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 0 \\ -2 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

So we see that the eigenvectors with eigenvalue 1 are of the form $(\lambda, -2\lambda, 2\lambda)$ and those with eigenvalue 2 are of the form $(\lambda, -\lambda, \lambda)$.

[3 marks] *Standard exercise.*

(d) In particular, the matrix M is not diagonalizable, since we can only find two linearly independent eigenvectors.

[1 mark] *Standard exercise.*

(e) The multiplicity of the eigenvalue 2 is two, and $(1, -1, 1)$ is an eigenvector of M for this eigenvalue. We need to find a vector $v = (a, b, c)$ such that $Mv = v_1 + 2v$. This is a linear equation: we have to solve

$$(M - 2I)v = v_1.$$

Writing this equation in matrix form, and transforming it into echelon form (doing the same transformations on $A - I$ as above), we get

$$\left(\begin{array}{rrr|r} 1 & 1 & 0 & 1 \\ -2 & -1 & 1 & -1 \\ 2 & 1 & -1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So $v = (a, b, c)$ is a suitable vector if and only if $a+b = 1$ and $b+c = 1$; for example, $v = (1, 0, 1)$ is a solution.

[5 marks] *Seen somewhat similar in exercises.*

So a basis which will put M into Jordan normal form is given by

$$(1, -1, 1), (1, 0, 1), (1, -2, 2).$$

To answer the question, we have to transform this basis back into our original vector space, where it becomes

$$B = (x^2 - x + 1, x^2 + 1, x^2 - 2x + 2).$$

[2 marks] *Unseen.*

We have

$$\varphi(x^2 - x + 1) = 2x^2 - 2x + 2 = 2(x^2 - 2 + 1) + 0 \cdot (x^2 + 1) + 0 \cdot (x^2 - 2x + 2),$$
$$\varphi(x^2 + 1) = 3x^2 - x + 3 = 1 \cdot (x^2 - x + 1) + 2 \cdot (x^2 + 1) + 0 \cdot (x^2 - 2x + 2),$$
$$\varphi(x^2 - 2x + 2) = 0 \cdot (x^2 + 1) + 0 \cdot (x^2 - x + 1) + 1 \cdot (x^2 - 2x + 2).$$

So the matrix of φ with respect to B is indeed

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

as expected.

[2 marks] *Standard.*

20 marks in total for Question 3

4.

(a) A *group* is a set G together with a binary operation $*$ such that: **(G1)** for all $g_1, g_2 \in G$, $g_1 * g_2 \in G$; **(G2)** for all $g_1, g_2, g_3 \in G$, $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$; **(G3)** there exists an element $e \in G$ such that, for all $g \in G$, $e * g = g * e = g$; **(G4)** for every $g \in G$, there exists $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$.

[1 marks]. *Standard definition from lectures.*

If G, H are groups, then a map $\varphi : G \to H$ is a *homomorphism* if, for all $g_1, g_2 \in G$, $\varphi(g_1 *_1 g_2) = \varphi(g_1) *_2 \varphi(g_2)$, where $*_1$ is the group law in G and $*_2$ is the group law in H.

[1 mark]. *Standard definition from lectures.*

The map φ is *injective* if, for all $g_1, g_2 \in G$, $\varphi(g_1) = \varphi(g_2) \Rightarrow g_1 = g_2$. The map φ is *surjective* if, for all $h \in H$, there exists $g \in G$ such that $\varphi(g) = h$.

[1 mark]. *Standard definitions from lectures.*

(b) Let $g_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$ be arbitrary elements of G. We have

$$\begin{aligned} \varphi(g_1 + g_2) &= 3((a_1 + a_2) + (b_1 + b_2)) - 6((c_1 + c_2) - (d_1 + d_2)) \\ &= 3a_1 + 3a_2 + 3b_1 + 3b_2 - 6c_1 - 6c_2 + 6d_1 + 6d_2 \\ &= (3(a_1 + 3b_1) - 6(c_1 - d_1)) + (3(a_2 + 3b_2) - 6(c_2 - d_2)) \\ &= \varphi(g_1) + \varphi(g_2). \end{aligned}$$

Hence φ is a homomorphism.

[2 marks]. *Seen similar in exercises.*

We have e.g.

$$\varphi \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 3 = \varphi \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

so φ is not injective. For any matrix $A \in G$, the value $\varphi(A)$ will be an integer multiple of 3. In particular, $\varphi(A) \neq 1$ for all $A \in G$, so φ is not surjective.

[2 marks]. *Seen similar in exercises.*

(c) Statements (i) and (iii) are true. Statement (ii) is false when the group is non-abelian; an example is given e.g. by taking the symmetric group $\mathcal{S}_3$.

[3 marks]. *From Lectures.*

Similarly, a counterexample in $\mathcal{S}_3$ to (iv) is given by letting a be the permutation which exchanges the first two elements, and b and c be the two cyclic permutations.

[2 marks]. *Unseen.*

(d) First of all, since $A = AC$, C must be the identity element of the group. So we can fill in the corresponding column and row:

*	A	B	C	D	E	F
A	B	?	A	E	?	?
B	C	?	B	?	?	?
C	A	B	C	D	E	F
D	?	E	D	C	?	?
E	?	?	E	A	?	?
F	?	?	F	?	?	C

Next, note that $BA = AAA = AB = C$. Furthermore, $BB = AAAA = AC = A$.

*	A	B	C	D	E	F
A	B	C	A	E	?	?
B	C	A	B	?	?	?
C	A	B	C	D	E	F
D	?	E	D	C	?	?
E	?	?	E	A	?	?
F	?	?	F	?	?	C

Every line and column in the group table must contain each element. The first row is only missing elements D and F; however, the last column already contains an F. So we can complete this row:

*	A	B	C	D	E	F
A	B	C	A	E	F	D
B	C	A	B	?	?	?
C	A	B	C	D	E	F
D	?	E	D	C	?	?
E	?	?	E	A	?	?
F	?	?	F	?	?	C

Similarly, we can fill in the second row, which is still missing D, E and F. Continuing in this way, we fill in the remaining entries:

*	A	B	C	D	E	F
A	B	C	A	E	F	D
B	C	A	B	F	D	E
C	A	B	C	D	E	F
D	F	E	D	C	B	A
E	D	F	E	A	C	B
F	E	D	F	B	A	C .

[5 marks]. *Seen somewhat similar in exercises.*

(e) The permutation group $\mathcal{S}_3$ has the same group table (letting C be the identity, A and B the two cyclic permutations, and D,E and F the permutations which keep one element fixed while switching the other two).

[3 marks]. *Unseen.*

20 marks in total for Question 4

5.

(a) The *rank* of φ is the dimension of $\mathrm{Im}(\varphi)$. The *nullity* of φ is the dimension of $\ker(\varphi)$.

[1 mark]. *Standard definitions from lectures.*

The rank and nullity theorem states that

$$\dim V = \mathrm{rank}(\varphi) + \mathrm{nullity}(\varphi).$$

[1 mark]. *Standard theorem from lectures.*

For $v_1 = (x_1, y_1, z_1)$ and $v_2 = (x_2, y_2, z_2)$ in $\mathbb{R}^3$ and $\lambda, \mu \in \mathbb{R}$, we have

$$\begin{aligned}
&\varphi(\lambda v_1 + \mu v_2)\\
&= \begin{pmatrix} ((\lambda x_1 + \mu x_2) + (\lambda y_1 + \mu y_2) + (\lambda z_1 + \mu z_2)) & (\lambda z_1 + \mu z_2) + (\lambda y_1 + \mu y_2) \\ 2(\lambda x_1 + \mu x_2) - (\lambda y_1 + \mu y_2) - (\lambda z_1 + \mu z_2) & 0 \end{pmatrix}\\
&= \begin{pmatrix} \lambda(x_1 + y_1 + z_1) + \mu(x_2 + y_2 + z_2) & \lambda(z_1 + y_1) + \mu(z_2 + y_2) \\ \lambda(2x_1 - y_1 - z_1) + \mu(2x_2 - y_2 - z_2) & 0 \end{pmatrix}\\
&= \lambda\varphi(v_1) + \mu\varphi(v_2).
\end{aligned}$$

Thus φ is linear.

[2 marks]. *Standard exercise.*

There are several ways of determining the rank and nullity; usually we would want to use the rank and nullity theorem. For example, let $(x, y, z) \in \mathbb{R}^3$. Then $(x, y, z) \in \ker(\varphi)$ if and only if

$$x + y + z = 0, \quad z + y = 0 \quad \text{and} \quad 2x - y - z = 0,$$

which is clearly the case if and only if $z = -y$ and $x = 0$. So

$$\ker(\varphi) = \{(0, y, -y) : y \in \mathbb{R}\} = \mathrm{span}((0, 1, -1)).$$

So $\mathrm{nullity}(\varphi) = 1$. Consequently $\mathrm{rank}(\varphi) = \dim(\mathbb{R}^3) - \mathrm{nullity}(\varphi) = 2$.

[3 marks]. *Standard exercise.*

(*Remark:* We have $\mathrm{Im}(\varphi) = \left\{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} : 2a = c + 3b \right\}$.)

(b) We have

$$\begin{aligned}
\ker(\varphi) &= \{(x, y, z, w) : x + z = w, 2x = z + 2w, 4x + z = 4w\}\\
&= \{(x, y, z, w) : z = 0, x = w\} = \{(x, y, 0, x) : x, y \in \mathbb{R}^4\}.
\end{aligned}$$

A basis for this space is given by $(1, 0, 0, 1), (0, 1, 0, 0)$, so $\mathrm{nullity}(\varphi) = 2$.

[2 marks]. *Standard exercise.*

In particular, we see that $\mathrm{rank}(\varphi) = 4 - 2 = 2$, so we only need to find two linearly independent vectors in the image of φ. Two such vectors are given by $v_1 = \varphi(0, 0, 1, 0) = (1, -1, 1)$ and $v_2 = \varphi(1, 0, 0, 0) = (1, 2, 4)$. (It is easy to check that $\mathrm{Im}(\varphi) = \{(x, y, z) : z = 2x + y\}$, so any basis of this space gives a correct answer.)

[2 marks]. *Standard exercise.*

(c) To put φ into standard form, we start by extending the given basis of $\ker(\varphi)$ to a basis of $\mathbb{R}^4$, for instance to the basis

$$B = ((0,0,1,0),(1,0,0,0),(1,0,0,1),(0,1,0,0)).$$

We need to check that these are linearly independent, but for this choice of vectors, this is immediately obvious.

[3 marks].

Now we need to take the two vectors of B which are not in the kernel and compute their images:

$$\varphi(0,0,1,0) = (1,-1,1) \quad \text{and} \quad \varphi(1,0,0,0) = (1,2,4).$$

We extend these two vectors to a basis of $\mathbb{R}^3$, e.g. by taking

$$C = ((1,-,1,1),(1,2,4),(1,0,0)).$$

Again we need to check that this really is a basis of $\mathbb{R}^3$. We could either check that the three vectors are linearly independent. Alternatively, it is easy to check directly that $(1,0,0)$ is not in the image of φ.

[3 marks].

It remains to compute the matrix A: we have

$$\begin{aligned}
\varphi(0,0,1,0) &= (1,-1,1) = 1\cdot(1,-1,1) + 0\cdot(1,2,4) + 0\cdot(1,0,0),\\
\varphi(1,0,0,0) &= (1,2,4) = 0\cdot(1,-1,1) + 1\cdot(1,2,4) + 0\cdot(1,0,0),\\
\varphi(1,0,0,1) &= (0,0,0) = 0\cdot(1,-1,1) + 0\cdot(1,2,4) + 0\cdot(1,0,0),\\
\varphi(0,1,0,0) &= (0,0,0) = 0\cdot(1,-1,1) + 0\cdot(1,2,4) + 0\cdot(1,0,0).
\end{aligned}$$

So we indeed have

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

as required.

[3 marks]. *Similar example seen on exercise sheet.*

20 marks in total for Question 5

6.

(a) A function $\varphi : V \to V$ is an *isomorphism* if φ is linear, injective and surjective.
[2 marks]. *Standard definition from lectures.*

(b) The composition $\varphi \circ \psi$ of two isomorphisms is again an isomorphism. Indeed, we see that linearity holds:

$$\varphi(\psi(\lambda v_1 + \mu v_2)) = \varphi(\lambda\psi(v_1) + \mu\psi(v_2)) = \lambda\varphi(\psi(v_1)) + \mu\varphi(\psi(v_2)).$$

If $\varphi(\psi(v_1)) = \varphi(\psi(v_2))$, then $\psi(v_1) = \psi(v_2)$ by injectivity of φ, and thus $v_1 = v_2$ by injectivity of ψ. So $\varphi \circ \psi$ is injective.

Let $w \in V$. Then by surjectivity of φ, there is $v_1 \in V$ such that $\varphi(v_1) = w$. By surjectivity of ψ, there is $v \in V$ such that $\psi(v) = v_1$. Then $\varphi(\psi(v)) = \varphi(v_1) = w$, so $\varphi \circ \psi$ is surjective.

[4 marks].

Associativity is clearly satisfied. The neutral element is given by the identity map $\varphi(v) = v$. The inverse element of φ is given by its inverse φ^{-1}.
[3 marks]. *Similar examples seen in exercises and lecture.*

(c) If V is n-dimensional, the dimension of $L(V, V)$ is n^2. (We saw in lectures that $L(V, V)$ is isomorphic to the space of $n \times n$-matrices; an isomorphism is given by the function which takes a linear map to its representation with respect to a given basis.) **[3 marks]**. *Seen (once) in lecture.*

(d) The set of isomorphisms is not a subspace of $L(V, V)$, as it does not contain the zero element; i.e. the linear map $\varphi(v) = 0$. **[4 marks]**. *Unseen.*

(e) $L(V, V)$ is not a group with respect to composition, since its identity element would have to be the identity map $\varphi(v) = v$, but e.g. the zero map $\varphi(0) = 0$ does not have an inverse. **[4 marks]**. *Unseen.*

20 marks in total for Question 6