1. (a) Say what it means for $\{v_1, \ldots, v_k\}$ to *span* a vector space V.
Let U be the subspace of $\mathbf{R}^3$ spanned by

$$u_1 = (1,1,1), u_2 = (1,2,0), u_3 = (2,3,1).$$

Let W be the subspace of $\mathbf{R}^3$ spanned by

$$w_1 = (1,3,-1), w_2 = (1,4,-2), w_3 = (2,7,-3).$$

Show that $U = W$.

(b) Let $V = M_2(\mathbf{R})$ be the vector space of 2×2 matrices with real entries. Let

$$U = \{\left(\begin{smallmatrix} a & a+b \\ a+b & b \end{smallmatrix}\right) : a, b \in \mathbf{R}\}, \quad W = \{\left(\begin{smallmatrix} a & a-b \\ a-b & b \end{smallmatrix}\right) : a, b \in \mathbf{R}\}.$$

Show that U and W are subspaces of V. What are the dimensions of each of $U, W, U \cap W$ and $U + W$? Is it true or false that $U + W = U \oplus W$? [20 marks]

2. (a) Let V be the vector space of polynomials in x of degree at most 2 with coefficients in $\mathbf{R}$. Let the linear map $L : V \to V$ be defined by

$$L(a + bx + cx^2) = a + (b + c)x + (b + c)x^2.$$

[You need not show that L is linear.] Find M, the matrix representation of L with respect to the basis $\{1, x, x^2\}$. What are the eigenvalues and eigenvectors of M?

(b) (i) Let $f : V \to W$ be a linear map between two vector spaces V and W. Define the *rank* of f and the *nullity* of f. State the rank & nullity theorem.

(ii) Let $V = M_2(\mathbf{R})$, the vector space of 2×2 matrices with real entries, and let $M = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 0 \end{smallmatrix}\right)$ and $N = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Let $F : V \to V$ be the linear map defined by

$$F(A) = MA + AN.$$

Find the matrix of F with respect to the basis $\{E_1, E_2, E_3, E_4\}$, where E_1, E_2, E_3, E_4 are $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, respectively.

Find a basis for the image of F and a basis for the kernel of F. Find the rank of F and the nullity of F. Verify that the rank & nullity theorem holds in this case. [20 marks]

3. (a) Define the terms: *group, homomorphism, injective, surjective.*

Let G be the group of real numbers under addition. Let H be the group of 2×2 matrices with real entries, under matrix addition [you need not show that these are groups]. Let $\phi : G \to H$ be defined by

$$\phi(g) = \begin{pmatrix} 2g & g \\ 0 & 0 \end{pmatrix}.$$

Show that ϕ is a homomorphism. State, giving reasons, whether ϕ is injective. State, giving reasons, whether ϕ is surjective.

(b) (i) Let G be a group. Show that the identity element e in G is unique. Show that $\alpha * \beta = e \Rightarrow \beta * \alpha = e$, for any $\alpha, \beta \in G$,

(ii) Show that $\alpha * \beta = \alpha * \gamma \Rightarrow \beta = \gamma$, for any group G and any $\alpha, \beta, \gamma \in G$, Deduce that no element can be repeated in the same row inside a group table. Similarly show that no element can be repeated in the same column.

(iii) The following is a partially completed table. Fill in the missing entries in a manner compatible with (i),(ii) above. You must justify (entry by entry) why each choice of entry is the only one possible. Then find an example where associativity does not hold. Hence deduce that the below table cannot be completed to give a group table.

*	A	B	C	D	E	F
A	D	?	?	C	?	?
B	F	?	?	?	?	?
C	?	?	?	?	?	?
D	B	?	A	E	?	?
E	?	A	B	?	?	E
F	?	?	?	?	?	?

[20 marks]

4. (a) Let the transformations $\rho_{A,\alpha}$ and σ_ℓ of $\mathbf{R}^2$ be as follows. For any point A and angle α, let $\rho_{A,\alpha}$ denote rotation anticlockwise about A through angle α. For any line ℓ let σ_ℓ denote reflection in the line ℓ.

(i) Let ℓ and m be two lines which both pass through point A. Let α be the angle from ℓ to m. Show that $\sigma_m \sigma_\ell = \rho_{A,2\alpha}$.

(ii) Let s be any line through a point B, and let β be any angle. Use part (i) to find a line r through B such that $\rho_{B,\beta} = \sigma_s \sigma_r$ and a line t such that $\rho_{B,\beta} = \sigma_t \sigma_s$. Hence show that $\rho_{B,\beta}\sigma_s = \sigma_s \rho_{B,\beta} \iff \beta = 0, \pi$.

(b) Define what it means for a matrix to be *orthogonal*. Let $P = \left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right)$ and $Q = \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)$ be 2×2 matrices with real entries; show that $(PQ)^T = Q^T P^T$.

Show that the set of 2×2 orthogonal matrices with real entries is a group under matrix multiplication. [20 marks]

5. (a) Let f be the bilinear form on $\mathbf{R}^2$ defined by

$$f((x_1, x_2), (y_1, y_2)) = x_1y_1 + x_2y_1 + 2x_2y_2.$$

Let $u_1 = (1, -1), u_2 = (1, 2)$. Compute $f(u_1, u_1), f(u_1, u_2), f(u_2, u_1), f(u_2, u_2)$. Find the matrix A of f relative to the basis $\{u_1, u_2\}$. Find the matrix B of f relative to the basis $\{v_1, v_2\}$, where $v_1 = (2, 1), v_2 = (0, 3)$.

Find the change of basis matrix P from $\{u_1, u_2\}$ to $\{v_1, v_2\}$ and show that $B = P^T AP$.

(b) Consider the quadratic form

$$q(x, y, z) = x^2 + 3y^2 + 4z^2 - 2xy + 2yz.$$

Give the matrix A representing q with respect to the standard basis. Find a diagonal matrix D equivalent to A and the matrix P which describes the change of basis from the standard basis to the basis in which q is diagonal. What are the rank and signature of q? Describe geometrically the surface $q(x, y, z) = 2$. Draw a sketch of the surface. [20 marks]

6. (a) For a real number θ, let $A(\theta)$ denote the following 2×2 matrix:

$$A(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Show that $A(\theta_1 + \theta_2) = A(\theta_1)A(\theta_2)$. Show also that $A(\theta)^{-1} = A(-\theta)$. Show that if $0 < \theta < \pi$ then $A(\theta)$ has no real eigenvalues.

(b) Suppose $\{x_1, \ldots, x_n\}$ is a basis for a vector space V. Describe the dual space V^* to V and describe how to define addition and scalar multiplication on V^* [you need not prove that V^* is a vector space]. Define the dual basis $\{\phi_1, \ldots, \phi_n\}$ to $\{x_1, \ldots, x_n\}$ and prove that it is a basis for V^*.

Consider the following vectors in $V = \mathbf{R}^3$:

$$v_1 = (1, 1, 2), \quad v_2 = (2, 0, -1), \quad v_3 = (0, 2, -3).$$

Show that these give a basis for $\mathbf{R}^3$. Find the dual basis $\{\phi_1, \phi_2, \phi_3\}$ of V^*. Compute $\phi_1((-2, 1, 1))$, $\phi_2((-2, 1, 1))$ and $\phi_3((-2, 1, 1))$. [20 marks]