1. (i) A ticket for a journey costs £1. A traveller without a ticket will be charged £10 if tickets are checked. Obtain the outcome matrix for the traveller, taking account of buying or not buying a ticket and of being checked by the railway company or not.

What are the traveller's security levels and what pure strategy optimizes them?

(ii) A simple version of the game of 'nim' is played as follows. There are two players and, at the start, two piles on the table in front of them, one pile containing two matches and the other pile containing three matches. In turn the players take *one* match from either of the piles. The first player to empty either pile is the loser. Sketch a game tree.

Enumerate the pue strategies for each player (assume player A goes first). Identify the winning player and that player's winning pure strategy.

(iii) Define a saddle point for a two-person zero-sum game played with pure strategies, where A's outcome matrix has entry u_{ij} corresponding to strategy A_i for A and B_j for B.

Show that, if both (A_i, B_j) and (A_k, B_l) are saddle points, then $u_{ij} = u_{kl}$.

Find all the saddle points of the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

2. Define α and β for a two-player game and show that, for a zero sum game, $\alpha + \beta \leq 0$.

Solve the following two-player zero-sum games, giving the optimal strategy for each player and the value of the game:

(i)

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

(ii)

$$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -3 \end{pmatrix}$$

(iii)

$$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \end{pmatrix}$$

3. The language AB over the alphabet $\Sigma \equiv \{a, b\}$ is defined by the regular expression

$$a^*(a \vee b)a$$

(i) Give all words from AB of length 3 or less.

(ii) Give all words from the regular expression $a^* \vee (a^*ba)$ which are not in AB.

(iii) Give a recursive definition of AB.

(iv) Draw a transition diagram for a finite automaton which accepts only the words of AB (reading from the right and terminated by blank).

Consider now the languages X and Y over the alphabet $\Sigma \equiv \{a, b\}$ which contain the set of words of the form

$$\text{X}: \quad b^n a b^m$$

$$\text{Y}: \quad b^n a b^n$$

where $n,\ m$ are non-negative integers.

Give productions for each of these languages and discuss the classification of the grammar in each case.

Describe two machines, one which accepts only words of the language X and the other which accepts only words of the language Y. You may assume in each case that the machine starts at the rightmost element of the word and that the word is terminated by a blank at the left.

4. (i) Let Γ be the directed graph whose adjacency matrix is as shown below:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Thus there is a directed edge from a to c but no directed edge from c to a. Determine the number of directed walks of length 2 from d to b and write them down. Which of these are directed paths? Calculate the in-degree and the out-degree at each vertex of Γ. Find a directed Euler walk in Γ, and write down, in order, the vertices passed in this walk.

(ii) Carry out a critical path analysis for the activity given below. Find the minimum time required to complete the task. Write down the critical path as a sequence of the α_i's. For each α_i not in this path, determine the float time.

Activity	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}	α_{13}	α_{14}
Time	5	4	8	10	6	3	12	2	7	11	9	5	3	4
Prerequisites				α_2	α_2	α_2	α_1	α_1	α_3	α_3	α_1	α_5	α_5	α_{10}
							α_4	α_4	α_6	α_6	α_4	α_8	α_8	α_{11}
												α_9	α_9	α_{12}

5. (i) Prove that a disjunctive normal form for $\neg\big(p \longleftrightarrow (p\wedge q)\big)$ is $p\wedge\neg q$. Hence prove that a disjunctive normal form for

$$\big((q \longleftrightarrow (p \wedge q)) \longrightarrow p\big) \wedge (p \longrightarrow \neg q)$$

is $(p \wedge \neg q) \vee (\neg p \wedge q)$.

(ii) Consider the following three sentences:

(a) $(p \longrightarrow \neg q) \wedge (\neg r \longrightarrow r)$

(b) $(\neg q \vee r) \longrightarrow (p \wedge \neg r)$

(c) $(\neg p \longleftrightarrow \neg r) \vee q$

Prove that none of these sentences implies either of the others. Show also that each of the sentences is true under every assignment of truth values to the propositional variables making both of the other two sentences false.

6. (i) Using the method of sentence tableau, decide whether or not the following sequent is valid:

$$(\neg q \longrightarrow \neg p) \vee (r \longrightarrow q) \vdash (p \wedge r) \longrightarrow q$$

(ii) Use the method of sentence tableau to determine whether or not the following argument is valid:

$$\forall x\Big(A(x) \wedge \forall y(B(x,y) \longrightarrow \neg C(y))\Big)$$
$$\exists x\Big(A(x) \longrightarrow \forall y(\neg D(y) \longrightarrow B(x,y))\Big)$$
$$\vdash \forall x\Big(C(x) \longrightarrow D(x)\Big)$$

(iii) Write the following argument symbolically and decide whether or not it is valid.

It is not possible to fly with umbrellas when the wind blows from the North;
Mary Poppins arrived yesterday by her umbrella with the East wind;
therefore, the East wind is not the same as the Northerly wind.