1. (a) Explain briefly the connexion between distance-squared functions on a regular plane curve and contact of the curve with circles. Explain in particular how it is possible to find the vertices of a regular plane curve using distance-squared functions. [6 marks]

Let λ be a constant nonzero real number and let $\gamma(t) = (t^2, \lambda t + t^3)$. Show that γ is a regular plane curve. [2 marks]

Express the distance-squared function $f(t)$ on γ from $(a, b) \in \mathbf{R}^2$ in the form $f(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \ldots$, for suitable coefficients c_i depending on a, b and λ. Hence or otherwise show that γ has a vertex at the point $\gamma(0) = (0, 0)$. Show further that the centre of curvature at this vertex is $(\frac{1}{2}\lambda^2, 0)$ and that the vertex is a higher vertex if and only if $\lambda = -\frac{1}{2}$. [11 marks]

(b) Write down the height function in the direction $(a, b) \neq (0, 0)$ on the regular plane curve $\gamma(t) = (t^2,\ t + t^3)$. Show that γ has an inflexion, which is not a higher inflexion, for each of the values $t = \pm\frac{1}{\sqrt{3}}$. [6 marks]

2. (a) Let $\gamma : I \to \mathbf{R}^3$ be a unit speed space curve. Define T and κ and, assuming κ is never zero, define N, B and τ. Show that $B' = -\tau N$. [6 marks]

Let $\alpha(s) = (\frac{1}{2}\cos s, \frac{1}{2}\sin s, \frac{1}{2}s\sqrt{3})$. Show that α is unit speed and calculate T, N, B, κ and τ in terms of s. [6 marks]

(b) Let $\gamma(t) = (t^2 - t, t^3, 2t^4 - t^5)$. Show that $\gamma : \mathbf{R} \to \mathbf{R}^3$ is a regular space curve. Write down the height function h on γ in the direction $(0, 1, -1)$ and find all the values of t for which h has an A_k singularity at t for some $k \geq 1$. In each case state the corresponding value of k. [7 marks]

Explain the connexion between this calculation and measurement of the contact between γ and the plane $y - z = 0$ (using (x, y, z) as coordinates in $\mathbf{R}^3$). [6 marks]

3. (a) In each of the following cases, find a local diffeomorphism $h : \mathbf{R}, t_0 \to \mathbf{R}, 0$ such that $f(t) = f(t_0) \pm (h(t))^2$:

$(i)\ f(t) = 1 - t^2 + t^5 + t^6, \quad t_0 = 0; \qquad (ii)\ f(t) = t - 2t^2 + t^3, \quad t_0 = 1.$

In each case, state briefly why your function h is a local diffeomorphism. [10 marks]

(b) Let $\phi : \mathbf{R}^2 \to \mathbf{R}^2$ be given by $\phi(x, y) = (w, z) = (x + y^2, xy - y^3)$. Write down the Jacobian matrix J of ϕ and show that J has determinant zero if and only if $x = 5y^2$. [2 marks]

Verify that $\phi(\frac{1}{2}, \frac{1}{2}\sqrt{2}) = (1, 0)$ and find all the other points points (x, y) for which $\phi(x, y) = (1, 0)$. What does the Inverse Function Theorem say about local inverses ψ for ϕ, $\psi(w, z) = (x, y)$ being defined near $(w, z) = (1, 0)$? For the local inverse taking $(1, 0)$ to $(\frac{1}{2}, \frac{1}{2}\sqrt{2})$, calculate $\partial x/\partial z$ and $\partial y/\partial z$ at $(w, z) = (1, 0)$. [13 marks]

4. (a) Let $\gamma : \mathbf{R}^2 \to \mathbf{R}^3$ be a smooth map. State what is meant by saying that γ is an *immersion* at $(x_0, y_0) \in \mathbf{R}^2$. In the following cases, determine the points (x_0, y_0) at which γ *fails* to be an immersion.

(i) $\gamma(x, y) = (x, xy, y^2)$;

(ii) $\gamma(x, y) = \alpha(x) + yN(x)$, where α is a unit speed space curve with κ never zero, and N is the principal normal vector. [11 marks]

(b) Define the terms *regular point* and *regular value* as applied to a smooth map $f : \mathbf{R}^m \to \mathbf{R}^q$, where $m \geq q$. [3 marks]

Let $f : \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x, y) = x^2 - 2xy^2 + y^5$. Find the regular points of f. Deduce that $f^{-1}(0) - \{(0, 0)\}$ is, in a neighbourhood of any of its points, a parametrized 1-manifold. Parametrizing by x or y as appropriate, find the curvature at $(1, 1)$. [11 marks]

5. Let $f : \mathbf{R}^3 \to \mathbf{R}^2$ be defined by

$$f(x, y, z) = (x^2 + y^2 + z^2, x^2 + y^2 - 2x).$$

Let f_1 be f restricted to the domain $\mathbf{R}^3 - \{(2, 0, 0)\}$. Show that $(4, 0)$ is a regular value of f_1. What does the implicit function theorem say about $M = f_1^{-1}(4, 0)$? [6 marks]

Show that there are no points of M where the tangent line is parallel to the x-axis, and find all the points of M at which the tangent line is parallel to the y-axis. [7 marks]

Show that $f^{-1}(4, 0)$ (note f here, not f_1) is precisely the set of points $\gamma(t) = (2\cos^2 t, 2\sin t \cos t, 2\sin t)$ where $t \in \mathbf{R}$. Show further that γ is a regular space curve. Why does this not show that $f^{-1}(4, 0)$ is a parametrized 1-manifold in a neighbourhood of the 'missing' point $\gamma(0) = (2, 0, 0)$?

[12 marks]

6. Let $F(t, x, y) = x\cos t + y\sin t - g(t)$ where g is a smooth function. Thus for each fixed t, $F_t(x, y) = 0$ is the equation of a straight line in the plane.

(i) Show that the envelope of the family of lines $F = 0$ is the set of points

$$(x, y) = (g(t)\cos t - g'(t)\sin t, \quad g(t)\sin t + g'(t)\cos t).$$

[4 marks]

(ii) Show that the points of regression on the envelope are points (x, y) given by t where $g(t) + g''(t) = 0$, and that $f = F_{(x,y)}$ has type exactly A_2 at t if and only if $g'(t) + g'''(t) \neq 0$. Show that in these circumstances F always versally unfolds f. What can you deduce about the structure of the envelope at such A_2 points? [12 marks]

(iii) Suppose that $g(t) > 0$ for all t. Explain why the distance from the origin to the line $F_t(x, y) = 0$ is $g(t)$. In the special case where $g(t) = 3+2\sin t$ show that the envelope has no cusps and that the tangents to the envelope at points corresponding to t and $t + \pi$ are parallel and a constant distance apart, for all t. [9 marks]

7. Let α be a unit speed space curve (parameter t, say), with $\kappa(t) > 0$ for all t. Let $r > 0$ be a constant real number. Let

$$F(\mathbf{x}, t) = (\mathbf{x} - \alpha(t)) \cdot (\mathbf{x} - \alpha(t)) - r^2,$$

where $\mathbf{x} \in \mathbf{R}^3$. Explain why, for a fixed t, $F_t(\mathbf{x}) = \mathbf{0}$ is the equation of the sphere with radius r centred at $\alpha(t)$. [2 marks]

Show that the envelope of the family of spheres given by F consists of points

$$\mathbf{x} = \alpha(t) + \lambda N(t) + \mu B(t), \quad \text{where } \lambda^2 + \mu^2 = r^2.$$

[4 marks]

Find the condition for $\mathbf{x}$ to be a point of regression on the envelope. In particular, show that, if $r < \frac{1}{\kappa(t)}$ for all t, then there are no points of regression. [7 marks]

Let $\mathbf{x} = (x, y, z)$. By writing $\alpha(t) = (X(t), Y(t), Z(t))$ or otherwise calculate $\partial F/\partial x, \partial F/\partial y$ and $\partial F/\partial z$. Hence show that the rows of the 2-jet matrix with constants are $2(\mathbf{x} - \alpha(t))$, $-2T(t)$ and $-2\kappa(t)N(t)$ respectively. [5 marks]

Assume that $f = F_{\mathbf{x}}$ has type A_2 at t. Show that F always versally unfolds f at t. (You do not need to find the condition for f to have type A_2.) What can you deduce about the local structure of the envelope in this case? [3 marks]

Now assume that $f = F_{\mathbf{x}}$ has type A_3 at t. Find the condition for F to versally unfold f at t. (You do not need to find the condition for f to have type A_3.) What can you deduce about the local structure of the envelope in this case? [4 marks]