PAPER CODE NO.
MATH421

THE UNIVERSITY of LIVERPOOL

JANUARY 2007 EXAMINATIONS

MMath: Year 4

LINEAR DIFFERENTIAL OPERATORS IN MATHEMATICAL PHYSICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks will be awarded for complete answers to FIVE questions. Only the best 5 answers will be taken into account.

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1. Let $u(x_1, x_2)$ satisfy the Poisson equation in the first quadrant $\Omega = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$ of the plane Ox_1x_2:

$$\Delta u(x_1, x_2) + \delta(x_1 - 1)\delta(x_2 - 3) = 0, \quad (x_1, x_2) \in \Omega \tag{1}$$

(a) Find the solution of (1), which satisfies the Dirichlet boundary conditions on $\partial\Omega$

$$u(x_1, 0) = 0 \;\text{ when } x_1 \geq 0, \tag{2}$$

and

$$u(0, x_2) = 0 \;\text{ when } x_2 \geq 0. \tag{3}$$

[10 marks]

(b) Find the solution of (1), which satisfies the Neumann boundary conditions on $\partial\Omega$

$$\frac{\partial u}{\partial x_2}(x_1, 0) = 0 \;\text{ when } x_1 \geq 0, \tag{4}$$

and

$$\frac{\partial u}{\partial x_1}(0, x_2) = 0 \;\text{ when } x_2 \geq 0. \tag{5}$$

[10 marks]

2. (a) Let $U(\mathbf{x}) \in \mathcal{D}'(\mathbf{R}^2)$ be a distribution, which satisfies the following equation

$$\Delta U(\mathbf{x}) + \delta(\mathbf{x}) = 0, \quad \mathbf{x} \in \mathbf{R}^2, \tag{6}$$

where $\delta(\mathbf{x})$ is the Dirac delta function, and Δ is Laplace's operator. Assume that derivatives $\partial U/\partial x_i, i = 1, 2$, vanish as $x_1^2 + x_2^2 \to \infty$. Verify that $U(\mathbf{x})$ can be represented in the form

$$U(\mathbf{x}) = \frac{1}{2\pi} \log |\mathbf{x}|^{-1} + B, \tag{7}$$

where B is constant.

[10 marks]

(b) Let (r, θ) be polar coordinates, and let $u(r, \theta)$ be a solution of the Dirichlet problem in the interior of the unit disk

$$\Delta u(r, \theta) = 0, \quad r < 1, \tag{8}$$

with the boundary condition

$$u(1, \theta) = \phi(\theta), \quad 0 < \theta \leq 2\pi, \tag{9}$$

where $\phi(\theta)$ is a smooth function. Verify that the function $u(r, \theta)$ can be represented in the form

$$u(r, \theta) = \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{\phi(\theta')d\theta'}{1 + r^2 - 2r\cos(\theta - \theta')}. \tag{10}$$

[10 marks]

3. Let $B_a = \{\mathbf{x} \in \mathbf{R}^3 : |\mathbf{x}| < a\}$ be a ball of radius a in $\mathbf{R}^3$, and let $S_a = \partial B_a$ be its boundary (the sphere of radius a with the centre at the origin).

(a) Verify that Green's function $G(\mathbf{x}, \mathbf{y})$ in B_a for the operator $-\Delta$ has the form

$$G(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} - \frac{q}{4\pi|\mathbf{x}^* - \mathbf{y}|}, \tag{11}$$

where $q = a/|\mathbf{x}|$ and $\mathbf{x}^* = q^2\mathbf{x}$.

[15 marks]

(b) Derive that the limit value of $G(\mathbf{x}, \mathbf{y})$, as $\mathbf{x} \to \mathbf{0}$, is given by

$$G(\mathbf{0}, \mathbf{y}) = \frac{1}{4\pi|\mathbf{y}|} - \frac{1}{4\pi a}. \tag{12}$$

[5 marks]

4. The fundamental solution for the heat equation is defined as

$$G(x, t) = \frac{1}{2\sqrt{\pi t}} \exp(-\frac{x^2}{4t}). \tag{13}$$

Find the temperature $T(x,t)$ of a semi-infinite conducting rod, assuming that T satisfies the initial boundary value problem:

$$\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} = 0, \quad x > 0,\ t > 0, \tag{14}$$

$$T(x,0) = \begin{cases} 1, & \text{when } 0 < x < 2, \\ -2, & \text{when } x \geq 2. \end{cases} \tag{15}$$

$$\frac{\partial}{\partial x} T(0,t) = 0, \quad t > 0. \tag{16}$$

[20 marks]

Hint: The error function is defined by

$$\text{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-\alpha^2} d\alpha. \tag{17}$$

Please also note that

$$\lim_{y \to +\infty} \text{erf}(y) = 1. \tag{18}$$

5. Let $D_R = \{(x_1, x_2) : x_1^2 + x_2^2 < R^2\}$ be a disk of radius R with the centre at the origin, and $\Xi(1,10) = D_{10} \setminus \overline{D_1}$ be a ring of the interior radius 1 and the exterior radius 10.

(a) Find a function $u(x_1, x_2)$, which satisfies the boundary value problem

$$\Delta u(x_1, x_2) = 0, \quad (x_1, x_2) \in \Xi(1,10), \tag{19}$$

$$u(x_1, x_2) = 1, \quad x_1^2 + x_2^2 = 100, \tag{20}$$

$$\frac{\partial u}{\partial n}(x_1, x_2) = 0, \quad x_1^2 + x_2^2 = 1, \tag{21}$$

where $\partial u / \partial n$ is the normal derivative of u on the unit circle.

[4 marks]

(b) Find a solution of the boundary value problem

$$\Delta u(x_1, x_2) = 0, \quad (x_1, x_2) \in \Xi(1,10), \tag{22}$$

$$u(x_1, x_2) = 1, \quad x_1^2 + x_2^2 = 100, \tag{23}$$

$$u(x_1, x_2) = 2, \quad x_1^2 + x_2^2 = 1. \tag{24}$$

[10 marks]

(c) For both cases (a) and (b), evaluate the energy integral

$$\mathcal{E}(u;\Xi(1,10)) = \int_{\Xi(1,10)} |\nabla u(x_1,x_2)|^2 dx_1 dx_2 \qquad (25)$$

[6 marks]

6. Consider the initial boundary value problem for the wave equation

$$\frac{\partial^2 u}{\partial x^2}(x,t) - 9\frac{\partial^2 u}{\partial t^2}(x,t) = 0, \quad x > 0,\ t > 0, \qquad (26)$$

with the boundary condition

$$u(0,t) = 0,\ t > 0, \qquad (27)$$

and the initial conditions

$$\frac{\partial u}{\partial t}(x,0) = x\exp(-x^2),\ x > 0, \qquad (28)$$

$$u(x,0) = 0,\ x > 0. \qquad (29)$$

(a) Determine the solution $u(x,t)$ of the initial boundary value problem (26)-(29).

[12 marks]

(b) Evaluate $f(t) = (\partial u/\partial x)(0,t)$. Prove that $f'(t) < 0$ for all $t > 3$.

[8 marks]

7. Within the elastic rod, $x \in (0, \pi)$, determine the displacement function $u(x,t)$, which satisfies the wave equation

$$\frac{\partial^2 u}{\partial x^2}(x,t) - \frac{1}{9}\frac{\partial^2 u}{\partial t^2}(x,t) = 0, \quad t > 0,\ 0 < x < \pi, \tag{30}$$

the boundary conditions

$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad \text{as } t > 0, \tag{31}$$

and the initial conditions

$$u(x,0) = 0,\ \frac{\partial u}{\partial t}(x,0) = x(\pi - x),\ 0 < x < \pi. \tag{32}$$

[20 marks]