1. (i) Let γ be a regular plane curve with parameter t. Explain briefly the two equations $\gamma' = s'\mathbf{T}$ and $\mathbf{T}' = \kappa s'\mathbf{U}$, where $'$ stands for d/dt. Write down a formula for $\mathbf{U}'$. Calculate γ'' and prove that

$$\kappa = \frac{[\gamma', \gamma'']}{||\gamma'||^3} \ .$$

(ii) Now let α be a unit speed curve and s a natural parameter on it. The *evolvent* of α with the *base value* $s = s_0$ (a fixed constant) of the parameter is the curve

$$\varepsilon(s) = \alpha(s) - (s - s_0)\mathbf{T}_\alpha(s)$$

where $\mathbf{T}_\alpha$ is the unit tangent to α.

Show that ε is regular unless either $s = s_0$ or $\kappa_\alpha(s) = 0$.

Assume from now on that $(s - s_0)\kappa_\alpha(s) > 0$. Show that $\mathbf{T}_\varepsilon(s) = -\mathbf{U}_\alpha(s)$ and $\mathbf{U}_\varepsilon(s) = \mathbf{T}_\alpha(s)$.

Find the curvature κ_ε of the evolvent.

Recall the expression for the centre of curvature of an arbitrary plane curve at a given point in terms of its curvature and unit normal. Find the centre of curvature of the evolvent ε at $\varepsilon(s)$.

The curve $\delta(s) = \varepsilon(s) + r\mathbf{U}_\varepsilon(s)$ is called *the parallel to ε at distance r*. Show that δ is also an evolvent of the initial curve α, but with some other base value s_1 of the parameter s. Find the relation between s_1, s_0 and r.

2. (i) Let $\gamma : I \to \mathbf{R}^3$ be a unit speed space curve. Define $\mathbf{T}$, κ, and assuming that $\kappa \neq 0$, define $\mathbf{P}$, $\mathbf{B}$ and τ. Show that

$$\mathbf{P}' = -\kappa\mathbf{T} + \tau B \qquad \text{and} \qquad \mathbf{B}' = -\tau\mathbf{P}\,.$$

Prove that $\qquad [\gamma', \gamma'', \gamma'''] = \kappa^2\tau\,.$

(ii) Show that a helix $\gamma : \mathbf{R} \to \mathbf{R}^3$, $\gamma(t) = \frac{1}{5}(\cos 4t, \sin 4t, 3t)$, is a unit speed curve and calculate $\mathbf{T}(t)$, $\mathbf{P}(t)$, $\mathbf{B}(t)$, $\kappa(t)$ and $\tau(t)$ for it.

3. Let $\mathbf{X} : U \to \mathbf{R}^3$ be a surface patch. Define the term *regular patch* and the coefficients E, F, G of the first fundamental form for $\mathbf{X}$.

Let $\mathbf{X}$ be a parametrisation

$$\mathbf{X}(u,v) = \big(f(u)\cos v, f(u)\sin v, u\big)$$

of the surface of revolution obtained by rotation of the graph $x = f(z)$ of a smooth function f about the z-axis. Calculate the coefficients of the first fundamental form of this surface and show that the surface is regular if f is never zero.

Consider a curve $\gamma(u) = \mathbf{X}(u, v(u))$, $u \in I \subset \mathbf{R}$, on our surface of revolution. Show that cosine of the angle α at which the curve γ intersects a meridian $v = \text{const}$ (oriented by the positive u direction) is given by the formula

$$\frac{\sqrt{E}}{\sqrt{E + Gv'^2}}$$

Assume that the curve γ meets all the meridians at the same angle α. Show that the interval of γ between two parallels $u = u_1$ and $u = u_2$ $(u_1 < u_2)$ has length

$$\frac{L}{\cos\alpha}$$

where L is the length of the part of any meridian bounded by the same parallels.

4. Define the coefficients e, f, g of the second fundamental form of a surface patch $\mathbf{X}$.

Let $\mathbf{X} : U \to \mathbf{R}^3$ be the graph surface

$$\mathbf{X}(u, v) = (u, v, h(u, v))$$

of a smooth function h defined on an open subset U of $\mathbf{R}^2$.

Show that this is a regular surface.

Find a unit normal $\mathbf{N}(u, v)$ to this surface and calculate the coefficients of the first and second fundamental forms. Deduce an expression for the Gaussian curvature of this surface (state without proof any general formula you use for K).

In the case when $h = u^3 - v^5$ and $U = \mathbf{R}^2$, show the subdivision of the (u, v)-plane into the sets of elliptic, hyperbolic and parabolic points of the surface. What are the asymptotic directions at the point $\mathbf{X}(1, -1)$?

5. Consider the tangent developable

$$\mathbf{X}(s,t) = \gamma(s) + t\gamma'(s)\,, \qquad t > 0,$$

of a unit speed space curve $\gamma : I \to \mathbf{R}^3$, $I \subset \mathbf{R}$. Assume that the curvature κ of γ is never zero.

Show that

(i) this is a regular parametrised surface, and the binormal $\mathbf{B}(s)$ of γ at $\gamma(s)$ can be taken for the unit normal of the surface at the point $\mathbf{X}(s,t)$;

(ii) the coefficients of the first fundamental form of $\mathbf{X}$ are

$$E = 1 + t^2\kappa^2\,, \qquad F = 1\,, \qquad G = 1\,;$$

(iii) the coefficients of the second fundamental form of $\mathbf{X}$ are

$$e = t\tau\kappa\,, \qquad F = 1\,, \qquad G = 1\,;$$

(iv) the principal curvatures at the point $\mathbf{X}(s,t)$ are

$$\kappa_1 = 0 \qquad \text{and} \qquad \kappa_2 = \frac{\tau}{t\kappa}\,;$$

(v) the principal curves on the surface are given by $s =$ const (these are the rulings of the developable) and $s + t =$ const.

6. (i) Consider a unit speed curve α on a surface M parametrised by a regular injective mapping $\mathbf{X} : U \to \mathbf{R}^3$, $U \subset \mathbf{R}^2$. Define the standard vectors $\mathbf{T}, \mathbf{N}, \mathbf{U}$ associated with α at the point $\alpha(s)$. Define the *normal (sectional)* and *geodesic curvatures* κ_n and κ_g of α at this point. In what case is α called a *geodesic*?

(ii) Consider a surface $M \subset \mathbf{R}^3_{x,y,z}$ parametrised by

$$\mathbf{X}(u,v) = (u, v, h(u,v)) \qquad \text{where} \quad h(0,0) = h_u(0,0) = h_v(0,0) = 0\,.$$

Let γ be a (non-unit speed) curve on M such that $\gamma(0) = (0,0,0)$ and $\gamma'(0) = (1,0,0)$. Show that at the origin

(a) the normal curvature of γ is equal, up to sign, to the ordinary curvature of the section $y = 0$ of M;

(b) the geodesic curvature of γ is equal, up to sign, to the ordinary curvature of its orthogonal projection to the (x,y)-plane.

(iii) Formulate Gauss' Theorem expressing a sectional curvature at a point of a surface in terms of the principal curvatures. Define the *mean curvature* $H(p)$ at a point p of a surface. Assuming $K(p) \neq 0$, show that $H(p) = 0$ if and only if the asymptotic directions at p are perpendicular to each other.

7. Explain the meaning of the term *Riemannian surface.*

Consider a Riemannian surface with coordinates u, v and the quadratic form having coefficients $E = 1$, $F = A(u)$, $G = B(v)$ (we assume that functions A and B on the surface are such that $B(v) > A^2(u)$).

Calculate the Christoffel symbols and the coefficients β_i^k using the formulae below.

Write out the Gauss-Weingarten equations for such a surface. Using these compute $(\mathbf{X}_{uu})_v$ and $(\mathbf{X}_{uv})_u$. Comparing the coefficients of $\mathbf{X}_v$ in the expressions obtained, deduce that the Gaussian curvature of the surface is

$$K = -\frac{A'B'}{2(B - A^2)^2}\,.$$

Deduce that K is identically zero if B is a constant function.

Let $\gamma : I \to \mathbf{R}^3$ be a unit speed curve, and c a constant vector. Define $\mathbf{X} : I \times \mathbf{R} \to \mathbf{R}^3$ by $\mathbf{X}(u, v) = \gamma(u) + vc$. Deduce from the above that the Gaussian curvature of $\mathbf{X}$ (at regular points) vanishes.

FORMULAE FOR QUESTION 7:

Christoffel symbols:

$$\begin{pmatrix} \Gamma_{11}^1 & \Gamma_{12}^1 & \Gamma_{22}^1 \\ \Gamma_{11}^2 & \Gamma_{12}^2 & \Gamma_{22}^2 \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} E_u/2 & E_v/2 & F_v - G_u/2 \\ F_u - E_v/2 & G_u/2 & G_v/2 \end{pmatrix}.$$

The β_i^k:

$$\begin{pmatrix} \beta_1^1 & \beta_2^1 \\ \beta_1^2 & \beta_2^2 \end{pmatrix} = -\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix}.$$

The Gauss-Weingarten equations:

$$\mathbf{X}_{uu} = \Gamma^1_{11}\mathbf{X}_u + \Gamma^2_{11}\mathbf{X}_v + e\mathbf{N}$$

$$\mathbf{X}_{uv} = \Gamma^1_{12}\mathbf{X}_u + \Gamma^2_{12}\mathbf{X}_v + f\mathbf{N}$$

$$\mathbf{X}_{vv} = \Gamma^1_{22}\mathbf{X}_u + \Gamma^2_{22}\mathbf{X}_v + g\mathbf{N}$$

$$\mathbf{N}_u = \beta^1_1\mathbf{X}_u + \beta^2_1\mathbf{X}_v$$

$$\mathbf{N}_v = \beta^1_2\mathbf{X}_u + \beta^2_2\mathbf{X}_v$$