1. Let $\gamma : I \to \mathbf{R}^2$ be a regular plane curve with unit tangent $\mathbf{T}$, unit normal $\mathbf{U}$ and curvature κ. Let r be a real number. The *parallel curve* δ to γ at distance r is the curve defined by

$$\delta(t) = \gamma(t) + r\mathbf{U}(t).$$

(i) In this part, you may assume that γ is unit speed. Show that δ is a regular curve except for values of t where $\kappa(t) \neq 0$ and $r = 1/\kappa(t)$.

Assume now that $r\kappa(t) < 1$ for all $t \in I$. Show that the unit tangent $\mathbf{T}_\delta$ and unit normal $\mathbf{U}_\delta$ to δ are the same vectors as $\mathbf{T}$ and $\mathbf{U}$ respectively. Show further that the curvature κ_δ of δ is given by

$$\kappa_\delta = \frac{\kappa}{1 - r\kappa}.$$

Show that the evolute of δ (namely, the curve $\delta + \mathbf{U}_\delta/\kappa_\delta$) coincides with the evolute $\gamma + \mathbf{U}/\kappa$ of γ.

(ii) For the (non unit speed) curve $\gamma(t) = (t, t^2)$ find an explicit parametrization of δ. Also find the curvature κ of γ and show that, for $r > \frac{1}{2}$, the parallel curve δ has exactly two points of non-regularity.

2. Let $\gamma : I \to \mathbf{R}^3$ be a regular space curve.

(i) Supposing γ is unit speed, and γ'' is never zero, define the standard vectors $\mathbf{T}$, $\mathbf{P}$, $\mathbf{B}$ and the curvature κ and torsion τ of γ. Prove that

$$(\gamma' \times \gamma'') \cdot \gamma''' = \kappa^2 \tau.$$

(ii) For a general regular space curve γ with $\kappa \neq 0$, write down formulae for the curvature and torsion of γ. Let u be a real constant and let

$$\gamma(t) = (t, t^3, t^4 + ut^2).$$

Show that, for $u \neq 0$, γ has no points where $\kappa = 0$. Show that for $u > 0$ there are exactly two points where the torsion is zero and for $u < 0$ there are no torsion zero points. Show that, for $u > 0$, binormal vectors at the two torsion zero points are parallel to

$$(-2u^{3/2}, \quad -12u^{1/2}, \quad \pm 3\sqrt{6}).$$

3. Let $\mathbf{X} : U \to \mathbf{R}^3$ be a surface patch. Define the term *regular* and the coefficients E, F, G of the first fundamental form for $\mathbf{X}$.

Let $\mathbf{X}$ be the surface patch

$$\mathbf{X}(u, v) = (u \cos v, \ u \sin v, \ u), \qquad u > 0.$$

Show that $\mathbf{X}$ is a regular patch. Draw a sketch of the surface M determined by $\mathbf{X}$. Show that $E = 2, \ F = 0$ and $G = u^2$.

Let $k > \sqrt{2}$ be constant, and let

$$\beta(u) = (u, \sqrt{k^2 - 2} \ln u), \qquad 1 \leq u \leq 3,$$

be a curve in the (u, v) plane. (Here ln is the natural logarithm.) Find the length of the corresponding curve $\gamma(u) = \mathbf{X}(\beta(u))$ on M.

Calculate a unit tangent to γ and show that this tangent makes a constant angle θ with the direction $(0, 0, 1)$, where $\cos\theta = 1/k$.

4. Let $\mathbf{X}$ be a parametrization of the surface of revolution obtained by rotating the regular curve $\alpha(u) = (\alpha_1(u), 0, \alpha_2(u))$ about the z-axis, namely

$$\mathbf{X}(u,v) = (\alpha_1(u)\cos v,\ \alpha_1(u)\sin v,\ \alpha_2(u)).$$

We shall assume that $\alpha_1(u)$ is never zero and that $\mathbf{X}$ is injective.

Find a unit normal $\mathbf{N}$ to this surface and calculate the coefficients of the first and second fundamental forms. Show that the principal directions at every point are given by $\mathbf{X}_u$ and $\mathbf{X}_v$.

Now let $\alpha(u) = (2+u^2, 0, u)$. Sketch this curve in the (x,z) plane. Show that the resulting surface of revolution has Gauss curvature

$$K = \frac{-2}{(2+u^2)(4u^2+1)^2}.$$

(State without proof any general formula you use for K.)

5. (i) Given a unit speed curve α on a (regular, injective) surface M parametrized by $\mathbf{X}$, define the three standard vectors $\mathbf{T}, \mathbf{N}, \mathbf{U}$ associated with α at the point $\alpha(s)$. Define the *geodesic curvature* κ_g of α at this point.

Explain briefly why, for any regular curve γ on M, and any non-zero vectors $\widetilde{\mathbf{T}}, \widetilde{\mathbf{U}}$ in the directions $\mathbf{T}$, $\mathbf{U}$ respectively,

$$\kappa_g = 0 \Leftrightarrow \widetilde{\mathbf{T}}' \cdot \widetilde{\mathbf{U}} = 0.$$

(ii) Now let $\mathbf{X}(s,t) = (x(s), y(s), t)$ be a surface patch, where x and y are functions of s with

$$x'^2 + y'^2 = 1$$

for all s. Show that $\mathbf{X}$ is regular. You may assume that $\mathbf{X}$ is injective.

Let k be a real constant. Consider the curve

$$\gamma(s) = \mathbf{X}(s, ks) = (x(s), y(s), ks),$$

on the surface M parametrized by $\mathbf{X}$. Find suitable vectors $\widetilde{\mathbf{T}}, \widetilde{\mathbf{U}}$ for γ and show that $\kappa_g = 0$ for all s, that is, γ is a geodesic.

Sketch the surface and the geodesic γ for the special case $x(s) = \cos s, y(s) = \sin s, k = 1$.

6. (i) Let M be a (regular, injective) surface with parametrization $\mathbf{X}$, and let α be a unit speed curve on M. What does it mean to say that the unit tangent $\mathbf{T}$ to α at a point $\mathbf{p}$ is (a) a *principal direction* at $\mathbf{p}$, (b) an *asymptotic direction* at $\mathbf{p}$? What is an *asymptotic curve* on M?

(ii) Let $\mathbf{X}(u,v) = (u, v, u^3 - 3uv^2)$, which is a parametrization of the 'Monkey saddle' M. Find a (non-unit) normal $\widetilde{\mathbf{N}}$ to M.

Consider a curve $\gamma(v) = \mathbf{X}(u(v), v)$ on M, where $u(v)$ is a smooth function of v and $u(0) = 0$. Show that

$$\widetilde{\mathbf{T}} = (u',\ 1,\ (3u^2 - 3v^2)u' - 6uv),$$

where the prime means d/dv, is a (non-unit) tangent vector to γ and that γ is an asymptotic curve on M if and only if

$$uu'^2 - 2u'v - u = 0$$

for all v. (State without proof the criterion you use for a curve to be asymptotic.)

By differentiating the last equation with respect to v, or otherwise, show that there are exactly three values for $u'(0)$.

7. (i) Let $\alpha : I \to \mathbf{R}^3$ be a regular space curve, and let β be a smooth family of unit vectors with β' never zero. Let

$$\mathbf{X}(s,t) = \alpha(s) + t\beta(s)$$

be the corresponding ruled surface.

Show that there is a unique function $r(s)$ such that, with $\gamma(s) = \alpha(s) + r(s)\beta(s)$, we have $\gamma'(s) \cdot \beta'(s) = 0$ for all s. (Thus γ is the *line of striction* on the ruled surface.)

Now let $\alpha(s) = (\cos s, \sin s, 0)$. Show that, for all s, the straight line through $\alpha(s)$, in the direction of the unit vector

$$\beta(s) = \frac{1}{\sqrt{2}}(-\sin s, \cos s, 1),$$

lies on the surface with equation $x^2 + y^2 - z^2 = 1$. Show that, for this ruled surface, α itself is the line of striction.

(ii) Let $\mathbf{X}$ be a (regular) surface. Let U be a region in the parameter plane. Write down the area of the region $\mathbf{X}(U)$ and deduce that $\mathbf{X}$ preserves area provided $EG - F^2 = 1$.

Verify that

$$\mathbf{X}(u,v) = (\sqrt{1-v^2}\cos u,\ \sqrt{1-v^2}\sin u,\ v), \quad -1 < v < 1$$

preserves area. Show that the image of this $\mathbf{X}$ is the unit sphere apart from the points $(0, 0, \pm 1)$.

Show that $\mathbf{X}$ may be obtained as follows. First map (u, v) to the cylinder $x^2 + y^2 = 1$ by the parametrization $\mathbf{Y}(u,v) = (\cos u, \sin u, v)$. Then join $\mathbf{Y}(u, v)$ to $(0, 0, v)$ by a straight line and take the intersection point with the unit sphere as $\mathbf{X}(u, v)$.

8. Let $\mathbf{X} : U \to \mathbf{R}^3$ be a parametrization of a (regular, injective) surface for which the first fundamental form has coefficients $E = A(u), F = 0, G = B(u)$ for some smooth positive functions A, B. So the first fundamental form is independent of the parameter v. Check that

$$\Gamma^1_{11} = E_u/2E,\ \Gamma^1_{12} = 0,\ \Gamma^1_{22} = -G_u/2E,$$

$$\Gamma^2_{11} = 0,\ \Gamma^2_{12} = G_u/2G,\ \Gamma^2_{22} = 0,$$

and give the corresponding expressions for the β^j_i in terms of E, G, e, f, g.

Write out the Gauss-Weingarten equations for such a surface, and use them to compute $(\mathbf{X}_{uu})_v$ and $(\mathbf{X}_{uv})_u$. Using the fact that the resulting expressions are equal, equate the coefficients of $\mathbf{X}_v$ to deduce that the Gauss curvature of the surface is given by

$$K = \frac{E_u G_u G - 2EGG_{uu} + EG_u^2}{4E^2G^2}.$$

Now consider the Riemannian surface M determined by

$$U = \{(u, v) \in \mathbf{R}^2 : u > 0\}$$

(the upper half plane) together with the first fundamental form given by $E = G = 1/u^2, F = 0$. Show that the Gauss curvature of this surface is -1.

The Gauss-Weingarten equations

$$\mathbf{X}_{uu} = \Gamma^1_{11}\mathbf{X}_u + \Gamma^2_{11}\mathbf{X}_v + e\mathbf{N}$$
$$\mathbf{X}_{uv} = \Gamma^1_{12}\mathbf{X}_u + \Gamma^2_{12}\mathbf{X}_v + f\mathbf{N}$$
$$\mathbf{X}_{vv} = \Gamma^1_{22}\mathbf{X}_u + \Gamma^2_{22}\mathbf{X}_v + g\mathbf{N}$$
$$\mathbf{N}_u = \beta^1_1\mathbf{X}_u + \beta^2_1\mathbf{X}_v$$
$$\mathbf{N}_v = \beta^1_2\mathbf{X}_u + \beta^2_2\mathbf{X}_v$$

The coefficients $\beta_i^j,\ \Gamma_{ij}^k$, are given by

$$\beta_1^1 = \frac{Ff - Ge}{EG - F^2},\ \beta_1^2 = \frac{Fe - Ef}{EG - F^2},\ \beta_2^1 = \frac{Fg - Gf}{EG - F^2},\ \beta_2^2 = \frac{Ff - Eg}{EG - F^2}$$

$$\Gamma_{11}^1 = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)},\ \Gamma_{12}^1 = \frac{GE_v - FG_u}{2(EG - F^2)},\ \Gamma_{22}^1 = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)},$$

$$\Gamma_{11}^2 = \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)},\ \Gamma_{12}^2 = \frac{EG_u - FE_v}{2(EG - F^2)},\ \Gamma_{22}^2 = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}.$$