MATH348 Solutions

October 13, 2003

1.
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx.$$

[2 marks]

Now let
$$f(x) = \frac{1}{(x^2+9)^2}.$$

To compute the Fourier transform, we consider the function
$$f(z) = \frac{e^{-iz\xi}}{(z^2+9)^2}.$$

[2 marks]

Let $\xi \geq 0$. If $\mathrm{Im}(z) \leq 0$ then $|e^{-iz\xi}| = e^{\mathrm{Im}(z)\xi} \leq 1$. So let $\gamma_R = \gamma_1(R) \cup \gamma_2(R)$ be the anticlockwise contour in the lower half plane, with $\gamma_1(R)$ being the straightline from R to $-R$ and $\gamma_2(R)$ being the semicircle arc. We have $|z^2+9| \geq |z|^2 - 9$. So
$$|f(z)| \leq \frac{1}{(R^2-9)^2} \text{ for } z \in \gamma_2(R).$$

[4 marks]

So
$$|\int_{\gamma_2(R)} \frac{e^{-iz\xi}}{(z^2+9)^2}dz| \leq \frac{\pi R}{(R^2-9)^2} \to 0 \text{ as } R \to \infty.$$

[2 marks]

We have
$$(z^2+9) = (z-3i)(z+3i) = 0$$
if and only if $z = 1 \pm i$. So the only singularity of f inside γ_R is at $-3i$. So
$$\int_{\gamma_R} f(z)dz = 2\pi i \mathrm{Res}(f(z), -3i) = \frac{d}{dz}((z-3i)^{-2}e^{-i\xi z})_{z=-3i}$$
$$= 2\pi i(-2(z-3i)^{-3}e^{-i\xi z} - i\xi(z-3i)^{-2}e^{-i\xi z})_{z=-3i}$$

$$= 2\pi i \left(\frac{-i}{108} + \frac{i\xi}{36} \right) e^{-3\xi} = \frac{\pi}{54}(1 - 3\xi)e^{-3\xi}.$$

[4 marks]

So

$$\hat{f}(\xi) = -\lim_{R \to \infty} \int_{\gamma_1(R)} f(z)dz$$

$$= -\lim_{R \to \infty} \int_{\gamma(R)} f(z)dz = \frac{\pi}{54}(1 - 3\xi)e^{-3\xi}.$$

[2 marks]

Now since $f(x)$ is real for real x,

$$\hat{f}(-\xi) = \int_{-\infty}^{\infty} f(x)e^{ix\xi}dx$$

$$= \overline{\int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx} = \overline{\hat{f}(\xi)}.$$

[3 marks]

So for all real ξ, we have

$$\hat{f}(\xi) = \frac{\pi}{54}(1 - 3|\xi|)e^{-3|\xi|}.$$

[1 mark]

2 + 2 + 4 + 2 + 4 + 2 + 3 + 1 = 20 marks. Similar to homework exercises. Of course, the very first definition is standard theory.

2(i) a) Define $f_n(x) = \frac{1}{x}\chi_{[1,n]}(x)$, where χ_A denotes the characteristic function $\chi_A(x) = 1$ for $x \in A$, $\chi_A(x) = 0$ otherwise. Then by the Fundamental Theorem of Calculus

$$\int f_n = [\log x]_1^n = \log(n).$$

We have $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [1,\infty)$ and $f_n(x) \le f_{n+1}(x)$ for all $x \in [1,\infty)$ and for all integers $n \ge 1$. By Monotone Convergence,

$$\int_0^\infty f(x)dx = \lim_{n\to\infty}\int f_n = \lim_{n\to\infty}\log(n) = +\infty.$$

So f is not integrable.

[3 marks]

b) Write $f_n = f\chi_{[-n,n]}$. Then by the Fundamental Theorem of Calculus

$$\int f_n = \int_0^n (x+1)^{-2}dx + \int_{-n}^0 (1-x)^{-2}dx$$

$= [-(x+1)^{-1}]_0^n + [(1-x)^{-1}]_{-n}^0 = 1-(n+1)^{-1}+1-(1+n)^{-1} = 2(1-(1+n)^{-1})$.
Again, $\lim_{n\to\infty} f_n(x) = f(x)$ for al x and for all $n \ge 1$, and since $f(x)$ is positive, $f_n(x) \le f_{n+1}(x)$ for all x and for all integers $n \ge 1$. So by Monotone Convergence

$$\int f = \lim_{n\to\infty}\int f_n = \lim_{n\to\infty} 2(1-(n+1)^{-1}) = 2 < +\infty.$$

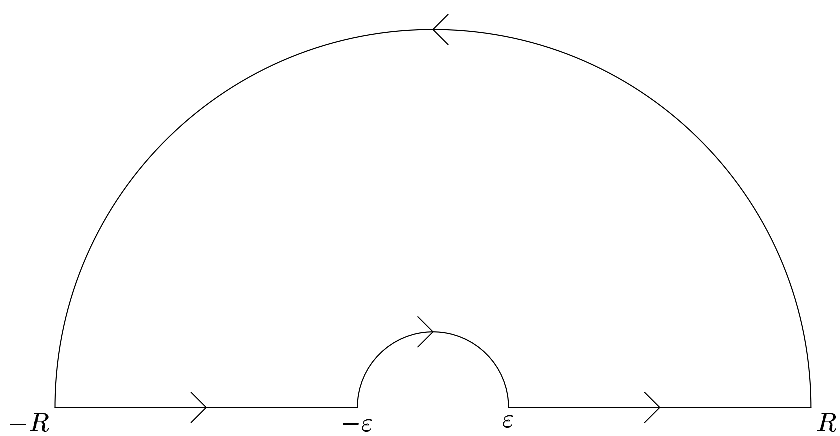
So f is integrable.

[4 marks]

c) Since $|\sin(x)| \le |x|$ for all x, we have $|f(x)| \le 1$ for all x. Also f is a continuous function on the given domain (and hence measuable). So, since 1 is integrable on $(0,1)$, f is too.

[3 marks]

(ii) Now consider the "beehive contour"$\gamma_{R,\epsilon}$ drawn.



Let γ'_R and γ'_ε be the semicircular parts of the contour. The function $\frac{e^{iz}}{z}$ has no singularities in or on $\gamma_{R,\epsilon}$. So by Cauchy's Theorem,

$$\int_{\gamma_{R,\epsilon}} \frac{e^{iz}}{z} dz = 0.$$

[2 marks]

$|e^{iz}| = e^{-\mathrm{Im}(z)} \leq 1$ on γ'_R. in fact $|e^{iz}| \leq e^{-\sqrt{R}}$ if $\mathrm{Im}(z) \geq \sqrt{R}$. So, since the length of γ'_R is πR,

$$\left| \int_{\gamma'_R} \frac{e^{iz}}{z} dz \right| \leq \frac{\pi\sqrt{R}}{R} + \frac{R\pi e^{-\sqrt{R}}}{R} \to 0 \text{ as } R \to \infty$$

[It is acceptable to simply say something like: the integral along γ'_R tends to 0 as $R \to \infty$ by Jordan's Lemma.]

[2 marks]

So

$$\lim_{R\to\infty,\varepsilon\to 0} \mathrm{Im} \left(\int_{-R}^{-\epsilon} + \int_{\varepsilon}^{R} \right) \frac{e^{ix}}{x} dx = \lim_{\varepsilon \to 0} \mathrm{Im} \left(\int_{\gamma'_\varepsilon} \frac{e^{iz}}{z} dz \right)$$

$$= \lim_{\varepsilon\to 0} \mathrm{Im} \left(\int_0^\pi e^{i\varepsilon e^{i\theta}} \frac{i\varepsilon e^{i\theta}}{\varepsilon e^{i\theta}} d\theta \right) = \lim_{\varepsilon \to 0} \mathrm{Im} \left(\int_0^\pi i(1 + i\varepsilon e^{i\theta} + \cdots) d\theta \right) = \pi.$$

[3 marks]

So

$$\lim_{R\to\infty,\varepsilon\to 0} \left(\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} \right) \frac{\sin x}{x} dx = \pi.$$

Since $\frac{\sin x}{x}$ is even,

$$\lim_{R\to\infty,\varepsilon\to 0} \int_\varepsilon^R \frac{\sin x}{x} dx = \lim_{R\to\infty} \int_0^R \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

[3 marks]

$3 + 4 + 3 + 2 + 2 + 3 + 3 = 20$ marks. (i) is similar to homework exercises. (ii) is a standard contour integral proved in lectures 9and of course they also have had many contour integrals as exercises).

3(i) If $x < y < x + \pi$ then $-\pi < x - y < 0$, and $g(x - y) = -x + y - \pi$. If $x - \pi < y < x$ then $0 < x - y < \pi$ and $g(x - y) = -x + y + \pi$.
[3 marks]

So if, as usual, we write

$$s_n(y) = \frac{\sin((n + \frac{1}{2})y)}{2\pi \sin(\frac{1}{2}y)},$$

$$S_n(g)(x) = \int_{x-\pi}^{x+\pi} (-x + y) s_n(y) dy + \left(\int_{x-\pi}^{x} - \int_{x}^{x+\pi} \right) \pi s_n(y).$$

[1 mark]

Since the integral of s_n over any interval of length 2π is 1, we obtain

$$S_n(g)(x) = -x + \int_{x-\pi}^{x+\pi} y s_n(y) dy + \left(\int_{x-\pi}^{x} - \int_{x}^{x+\pi} \right) \pi s_n(y) dy.$$

[3 marks]

The Fourier Series Theorem says that for each x

$$\lim_{n\to\infty} S_n(g)(x) = \frac{1}{2}(g(x+) + g(x-)) = -x + \pi \text{ for } 0 < x < \pi.$$

[2 marks]

(ii) Make the change of variable $u = (n + \frac{1}{2})y$. Then $du = (n + \frac{1}{2})dy$ and $dy/y = du/u$. When $y = x_n$ then $u = \pi$ and when $y = x_n \pm \pi$, $u = \pi \pm (n + \frac{1}{2})\pi$. So

$$T_n(g)(x_n) = \left(\int_{\pi(\frac{1}{2}-n)}^{\pi} - \int_{\pi}^{\pi(n+1+\frac{1}{2})} \right) \frac{\sin u}{u} du.$$

[3 marks]

Now $(\sin u)/u$ is an even function and

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin u}{u} du = 2 \lim_{R\to\infty} \int_{0}^{R} \frac{\sin u}{u} du$$

exists. So

$$\lim_{n\to\infty} T_n(g)(x_n) = \lim_{R\to\infty} \int_{-R}^{\pi} \frac{\sin u}{u} du - \lim_{R\to\infty} \int_{\pi}^{R} \frac{\sin u}{u} du = 2 \int_{0}^{\pi} \frac{\sin u}{u} du.$$

[3 marks]

Now if convergence of $S_n(g)(x)$ to its limit is uniform then the limit is $-x+\pi$ for all $x \in (0, 2\pi)$, and given $\varepsilon > 0$, there is N such that for all $n \geq N$ and all $x \in (0, \pi)$,

$$|S_n(g)(x) + x - \pi| < \varepsilon.$$

But

$$\lim_{n\to\infty} T_n(g)(x_n) = 2 \int_{0}^{\pi} \frac{\sin y}{y} dy = \pi + a$$

for some $a > 0$. So taking $\varepsilon = \frac{1}{2}a$ and any n such that $x_n < \frac{1}{2}a$ we get a contradiction.
[5 marks]
$3 + 1 + 3 + 2 + 3 + 3 + 5 = 20$ marks. Similar to homework exercise.

4(i) We have

$$\hat{u}_x(n,t) = \int_0^{2\pi} u_x(x,t)e^{-inx}dx = [u(x,t)e^{-inx}]_0^{2\pi} + in\int_0^{2\pi} u(x,t)e^{-inx}dx$$

$$= 0 + in\hat{u}(n,t).$$

Similarly,

$$\hat{u}_{xx}(n,t) = in\hat{u}_x(n,t) = -n^2\hat{u}(n,t).$$

By differentiating under the integral sign

$$\hat{u}_t(n,t) = \frac{d}{dt}\hat{u}(n,t).$$

[4 marks]

So from (1) and (2) we derive

$$\frac{d}{dt}\hat{u} = -n^2\hat{u},$$

$$\hat{u}(n,0) = \hat{f}(n).$$

The solution is

$$\hat{u}(n,t) = \hat{f}(n)e^{-n^2t}.$$

So the Fourier series

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \hat{u}(n,t)e^{inx}$$

for $u(x,t)$ with respect to x becomes

$$\frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{-n^2t+inx}.$$

[3 marks]

(ii) Each of the functions $u(x,t)$ is piecewise smooth in x, for $x \in [0,2\pi]$, and extends to a 2π-periodic continuous and piecewise smooth function of $x \in \mathbf{R}$. [This in itself is enough to ensure that

$$\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < +\infty.]$$

So the standard Fourier Series Theorem says that

$$u(x,t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \hat{u}(n,t)e^{inx}.$$

So

$$\left|u(x,t) - \frac{1}{2\pi}\hat{f}(0)\right| \leq \sum_{n\neq 0, n=-\infty}^{\infty} |\hat{u}(n,t)|$$

$$= \sum_{n\neq 0, n=-\infty}^{\infty} |\hat{f}(n)| e^{-n^2 t}$$

$$\leq e^{-t} \sum_{n=-\infty}^{\infty} |\hat{f}(n)| \leq Ce^{-t}.$$

[5 marks]

Also, applying the Fourier Series Theorem to both f and $u(x,t)$ (in x) we have

$$|u(x,t) - f(x)| = \left| \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (e^{-n^2 t} - 1)\hat{f}(n) e^{inx} \right|$$

$$\leq \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} |e^{-n^2 t} - 1||\widehat{f}(n)|.$$

Now for all $y \geq 0$, $0 < e^{-y} \leq \text{Max}(1, y)$. So for any integer $N > 0$,

$$|u(x,t) - f(x)| \leq \frac{1}{2\pi} \left(\sum_{|n|\leq N} N^2 t |\hat{f}(n)| + \sum_{|n|>N} |\hat{f}(n)| \right)$$

$$\leq CN^2 t \sum_{n=-N}^{N} |\hat{f}(n)| + C \sum_{|n|>N} |\hat{f}(n)|$$

as required.

[5 marks]

Now given $\varepsilon > 0$, choose N so that

$$C \sum_{|n|>N} |\hat{f}(n)| < \frac{\varepsilon}{2}.$$

Then choose $\delta > 0$ so that

$$CN^2 \delta \sum_{n=-N}^{N} |\hat{f}(n)| < \frac{\varepsilon}{2}.$$

Then if $0 < t < \varepsilon$ we have

$$|u(x,t) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which gives

$$\lim_{t\to 0} u(x,t) = f(x),$$

as required.

[3 marks]

$4 + 3 + 5 + 5 + 3 = 20$ marks.

5(i) We have

$$\lim_{R\to\infty}\int_{\gamma_3(R)} e^{-z^2/2}dz = -\lim_{R\to\infty}\int_{-R}^{R} e^{-(x+i\xi)^2/2}dx$$

$$= -\lim_{R\to\infty}\int_{-R}^{R} e^{-x^2/2+\xi^2/2-ix\xi}dx = -e^{\xi^2/2}\hat{f}(\xi).$$

[2 marks]

On $\gamma_2(R)$ we have $z = R + iy$ for $0 \le y \le \xi$. So $z^2 = R^2 + 2iRy - y^2$. Then $|e^{-z^2/2}| = e^{-R^2/2+y^2/2}$. So

$$\left|\int_{\gamma_2(R)} e^{-z^2/2}dz\right| \le \text{length}(\gamma_2(R))e^{-R^2/2+\xi^2/2} = \xi e^{-R^2/2+\xi^2/2} \to 0 \text{ as } R\to\infty.$$

Similarly if $z = -R + iy$ then $z^2 = R^2 - 2iRy - y^2$ and

$$\lim_{R\to\infty}\int_{\gamma_4(R)} e^{-z^2/2}dz = 0.$$

[3 marks]

Since $e^{-z^2/2}$ is holomorphic in the whole plane, we have

$$\int_{\gamma(R)} e^{-z^2/2}dz = 0.$$

Now

$$\hat{f}(\xi) = -\lim_{R\to\infty} e^{-\xi^2/2}\int_{\gamma_3(R)} e^{-z^2/2}dz = \lim_{R\to\infty} e^{-\xi^2/2}\int_{\gamma_1(R)} e^{-z^2/2}dz$$

$$= e^{-\xi^2/2}\int_{-\infty}^{\infty} e^{-x^2/2}dx = e^{-\xi^2/2}\sqrt{2\pi}.$$

Since we have $\hat{f}(-\xi) = \hat{f}(\xi)$ we have, for all ξ,

$$\hat{f}(\xi) = e^{-\xi^2/2}\sqrt{2\pi}.$$

[3 marks]

(ii) We have

$$\frac{\hat{f}(\xi+h)-\hat{f}(\xi)}{h} + i\hat{g}(\xi) = \int_{-\infty}^{\infty}\left(\frac{e^{-ix(\xi+h)}-e^{-ix\xi}}{h} + ixe^{-ix\xi}\right)f(x)dx$$

$$= \int_{-\infty}^{\infty}\left(\frac{e^{-ixh}-1}{h} + ix\right)e^{-ix\xi}f(x)dx.$$

So

$$\left|\frac{\hat{f}(\xi+h)-\hat{f}(\xi)}{h}+i\hat{g}(\xi)\right| \leq \int_{-\infty}^{\infty}\left|\frac{e^{-ixh}-1}{h}+ix\right||f(x)|dx$$

$$\leq 3\int_{-\infty}^{\infty}\mathrm{Min}(|x|,|h|x^2)|f(x)|dx,$$

using (ii) with $a = xh$.
[5 marks]

Splitting the integral up into $|x| \leq h^{-1/3}$ - where $|h|x^2 \leq |h|^{1/3}$ - and $|x| \geq |h|^{-1/3}$ - we have

$$\left|\frac{\hat{f}(\xi+h)-\hat{f}(\xi)}{h}+i\hat{g}(\xi)\right| \leq 3|h|^{1/3}\int_{-\infty}^{\infty}|f(x)|dx + 3\int_{|x|\geq h^{-1/3}}|g(x)|dx \to 0 \text{ as } h \to 0.$$

This shows that $\hat{f}$ is differentiable and

$$\frac{d}{d\xi}\hat{f}(\xi) = -i\hat{g}(\xi).$$

[3 marks]
(iii) If $g(x) = xe^{-x^2/2}$ and $h(x) = x^2e^{-x^2/2}$ then by the assumpton we can make, we have

$$\hat{g}(\xi) = i\frac{d}{d\xi}\hat{f}(\xi) = \sqrt{2\pi}(-i\xi)e^{-\xi^2/2}$$

[2 marks]

Similarly we have

$$\hat{h}(\xi) = i\frac{d}{d\xi}\hat{g}(\xi) = \sqrt{2\pi}\frac{d}{d\xi}(\xi e^{-\xi^2/2} = \sqrt{2\pi}(1-\xi^2)e^{-\xi^2/2}.$$

[2 marks]

$2+3+3+5+3+2+2 = 20$ marks. (i) is theory from lectures (but also an example of a contour integral). (ii) is guided theory from lectures, and (iii) an example which could be solved by a different method from that suggested here - and was actually set as a homework problem with a suggestion for solving in a different way.)

6.(i) *Tonelli's Theorem.* If one of

$$\int\int |f(x,y)|dxdy, \ \int\int |f(x,y)|dydx$$

is finite, then

$$x \mapsto \int f(x,y)dy, \ y \mapsto \int f(x,y)dx$$

are defined almost everywhere, and the double integrals

$$\int\int f(x,y)dxdy, \ \int\int f(x,y)dydx$$

are both defined and are equal.
[5 marks]
(ii)

$$(\widehat{f}\widehat{g})^{\vee}(y) = \int_{-\infty}^{\infty} \widehat{f}(\xi)\widehat{g}(\xi)e^{i\xi y}d\xi = \int_{-\infty}^{\infty} \widehat{g}(\xi)e^{i\xi y}\int_{-\infty}^{\infty} f(x)e^{-i\xi x}dxd\xi.$$

Now

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\widehat{g}(\xi)||e^{i\xi(y-x)}||f(x)|dxd\xi = \int_{-\infty}^{\infty} |\widehat{g}(\xi)|d\xi\int_{-\infty}^{\infty} |f(x)|dxd\xi < \infty.$$

So by Tonelli,

$$(\widehat{f}\widehat{g})^{\vee}(y) = \int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty} \widehat{g}(\xi)e^{i\xi(y-x)}d\xi dx = \int_{-\infty}^{\infty} f(x)(\widehat{g})^{v}(y-x)dx$$

$$= \int_{-\infty}^{\infty} f(y-t)(\widehat{g})^{\vee}(t)dt$$

(using the change of variable $t = y - x$, $dt = -dx$)

$$= f * (\widehat{g})^{\vee}(y).$$

[5 marks]
(iii) By change of variable $y/\sqrt{t} = u$,

$$\int_{|y|\geq\delta} \frac{1}{\sqrt{2\pi t}}e^{-y^2/2t}dy = \int_{|u|\geq\delta/\sqrt{t}} \frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy \to 0 \text{ as } t \to 0.$$

Similarly

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}}e^{-y^2/2t}dy = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy = 1.$$

[4 marks]
So

$$f(x) - \frac{1}{2\pi}(\widehat{f}\widehat{g}_\lambda)^{\vee}(x) = f(x) - f * g_t(x) = \int_{-\infty}^{\infty}(f(x)g_t(y) - f(x-y)g_t(y))dy.$$

[1 mark]

Let $|f(y)| \leq M$ for all y. Fix x. Given $\epsilon > 0$ choose $\delta > 0$ so that $|f(x) - f(x-y)| < \epsilon/2$ for all $|y| \leq \delta$. Then for this δ, choose t_0 so that

$$\int_{|y|\geq\delta} g_t(y)dy \leq \frac{\varepsilon}{4M} \text{ for all } t \leq t_0.$$

Then

$$\left|\int_{|y|\geq\delta} (f(x) - f(x-y))g_t(y)dy\right| \leq 2M \int_{|y|\geq\delta} g_t(y)dy < \frac{\varepsilon}{2} \text{ for all } t \leq t_0.$$

Then

$$\left|f(x) - \frac{1}{2\pi}(\widehat{f}\widehat{g}_t)^\vee(x)\right| \leq \frac{\varepsilon}{2}\int_{|y|\leq\delta} g_t(y)dy + \frac{\varepsilon}{2} \leq \varepsilon \text{ for all } t \leq t_0.$$

[5 marks]

$5 + 5 + 4 + 1 + 5 = 20$ marks.

7.(i)

$$\mathcal{L}(f)(z) = \int_0^\infty f(x)e^{-xz}dx.$$

[1 mark]

Write $z = t + iu$ for t and u real. Then

$$|e^{-xz}| = |e^{-xt-ixu}| = e^{-xt} \leq 1$$

for $x \geq 0$ and $t \geq 0$. So for $\mathrm{Re}(z) \geq 0$,

$$|\mathcal{L}(f)(z)| \leq \int_0^\infty |f(x)e^{-xz}|dx \leq \int_0^\infty |f(x)|dx = \|f\|_1,$$

and so $\mathcal{L}(f)(z)$ is bounded.
[3 marks]

$$|\mathcal{L}(f)(z)| \leq \left| \int_0^\infty f(x)e^{-xz}dx \right| \leq \int_0^\infty |f(x)||e^{-xz}|dx$$
$$= \int_0^\infty |f(x)|e^{-\mathrm{Re}(z)x}dx.$$

Now by Dominated Convergence (which works for functions parametrised by the positive reals), since $|f(x)| \geq e^{-Rx}|f(x)|$ and $|f(x)|$ is integrable and $\lim_{R\to\infty} e^{-Rx}|f(x)| = 0$ for all x,

$$\lim_{R\to\infty} \int_0^\infty |f(x)|e^{-Rx}dx = \int_0^\infty 0 = 0.$$

[4 marks]

Now we assume also that

$$\lim_{\mathrm{Im}(z)\to\infty} \mathcal{L}(f)(z) = 0$$

uniformly for $0 \leq \mathrm{Re}(z) \leq A$.

So given $\varepsilon > 0$, we can choose A so that if either $\mathrm{Re}(z) > A$ or $|\mathrm{Im}(z)| > A$ then

$$\left| \int_0^\infty f(x)e^{-xz}dx \right| \leq \varepsilon.$$

This says precisely that

$$\lim_{z\to\infty, \mathrm{Re}(z)>0} \mathcal{L}(f)(z) = 0.$$

[3 marks]
(ii) a) F_1 is not holomorphic for $\mathrm{Re}(z) > 0$ (in fact, not for z in any open set), and therfore cannot be $\mathcal{L}(f)(z)$ for any $f \in L^1(0, \infty)$.
[2 marks]
(ii) b) $F_2(re^{i\pi/4}) \not\to 0$ as $r \to +\infty$. So, by (i), F_2 cannot be $\mathcal{L}(f)(z)$ for any $f \in L^1(0, \infty)$.

[3 marks]
(ii) c) $F_3(z) = \mathcal{L}(f)(z)$ where $f(x) = e^{-x}$, because

$$\int_0^\infty e^{-x-zx}dx = \left[\frac{-1}{z+1}e^{-x(z+1)}\right]_0^\infty = \frac{1}{z+1}.$$

[2 marks]
(ii) d) $F_4(z)$ has a singularity at $z = 1$ and hence is not holomorphic for $\mathrm{Re}(z) > 0$, and therefore cannot be $\mathcal{L}(f)(z)$ for any $f \in L^1(0, \infty)$.
[2 marks]
$1 + 3 + 4 + 3 + 2 + 3 + 2 + 2 = 20$ marks. (i) is partly theory from lectures, and was partly a homework exercise. (ii) is similar to homework exercises.

8. (i) If we write m and σ for the mean and variance respectively of a probability measure μ, then

$$m = \int_{-\infty}^{\infty} d\mu$$

if x is integrable with respect to μ, and

$$\sigma = \int_{-\infty}^{\infty} (x-m)^2 d\mu$$

if x^2 is integrable with respect to μ - in which case x is too, and so m is defined. [3 marks]

Now

$$\int_{-\infty}^{\infty} e^{-|x|}dx = 2\int_0^{\infty} e^{-x}dx = \lim_{N\to\infty} 2[-e^{-x}]_0^N = 2.$$

So f is indeed the density for a probability measure μ. The function $x^2 f(x) \leq Ce^{-|x|/2}$. So both the mean and the variance of μ do exist. If we again call these m and σ: $xe^{-|x|}$ is an odd fnction. So we immediately see that $m = 0$. Then $x^2e^{-|x|}$ is an even function. So

$$\sigma = 2\int_0^{\infty} \frac{x^2e^{-x}}{2}dx = \lim_{N\to\infty}[-x^2e^{-x}]_0^N + 2\int_0^{\infty} xe^{-x}dx$$

$$= 2\lim_{N\to\infty}[-xe^{-x}]_0^N + 2\int_0^{\infty} e^{-x}dx = 2.$$

[4 marks]

(ii) We have

$$\hat{\mu}(\xi) = \hat{f}(\xi) = \frac{1}{2}\int_0^{\infty} e^{-ix\xi - x}dx + \frac{1}{2}\int_{-\infty}^{0} e^{-ix\xi + x}dx$$

$$= \frac{1}{2}\lim_{N\to\infty}[\frac{-e^{-ix\xi-x}}{1+i\xi}]_0^N + \frac{1}{2}\lim_{N\to\infty}[\frac{e^{-ix\xi+x}}{1-i\xi}]_{-N}^0$$

$$= \frac{1}{2(1+i\xi)} + \frac{1}{2(1-i\xi)} = \frac{1}{1+\xi^2}.$$

[4 marks]

We have

$$\widehat{(*)^n}\mu(\xi) = (\hat{\mu}(\xi))^n.$$

We also have

$$\hat{\mu}_n(\xi) = \int e^{-ix\xi/\sqrt{n}})d(*^n\mu) = \widehat{(*)^n}\mu(\xi/\sqrt{n})$$

$$= (\hat{\mu}(\xi/\sqrt{n}))^n = \left(\frac{1}{1+(\xi^2/n)}\right)^n$$

[3 marks]

So

$$\lim_{n\to\infty} \ln \hat{\mu}_n(\xi) = \lim_{n\to\infty} (-n\ln(1+(\xi^2/n)))$$
$$= \lim_{n\to\infty} (-n(\xi^2/n - \xi^4/2n^4 + \cdots) = -\xi^2.$$

[3 marks]
So

$$\lim_{n\to\infty} \hat{\mu}_n(\xi) = e^{-\xi^2}$$

which we are allowed to assume is the Fourier transform of the measure with density function $(1/\sqrt{\pi})e^{-x^2/4}$. This function has mean 0 (because it is an even function) and variance 2 because

$$\int_{-\infty}^{\infty} x^2 e^{-x^2/4} = \lim_{N\to\infty} [(-2x)e^{-x^2/4}]_{-N}^{N} + 2\int_{-N}^{N} e^{-x^2/4} dx.$$

The Central limit theorem says that

$$\lim_{n\to\infty} \int g(x) d\mu_n(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2/4} g(x)$$

In particular this should be true for $g(x) = e^{-ix\xi}$ -as it is.
[3 marks]
$3+4+4+3+3+3 = 20$ marks. First part of (i) is theory from lectures. The rest is similar to homework exercises.