Solutions to MATH348 May 2001

1.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-ix\xi} f(x)dx.$$

[2 marks]

If f is real-valued,

$$\overline{\hat{f}(\xi)} = \int_{-\infty}^{\infty} e^{ix\xi} f(x)dx = \hat{f}(-\xi).$$

[2 marks].

Now let

$$f(x) = \frac{1}{(x^2+1)^3}.$$

Let γ_R be the semicircular contour of radius R in the lower half plane, with curved piece γ_R'. Then

$$-\int_{-R}^{R} \frac{e^{-ix\xi}dx}{(x^2+1)^3} + \int_{\gamma_R'} \frac{dz}{(z^2+1)^3} = \int_{\gamma_R} \frac{e^{-i\xi z}dz}{(z^2+1)^3}$$

$$= 2\pi i \mathrm{Res}\left(\frac{e^{-i\xi z}}{(z^2+1))^3}, i\right)$$

by the residue formula. The zeros of $z^2+1 = (z+i)(z-i)$ are $\pm i$ and only $-i$ is inside γ_R (assuming $R > 1$. We have

$$\mathrm{Res}\left(\frac{e^{-i\xi z}}{(z^2+1)^3}, i\right) = \frac{1}{2!}\frac{d^2}{dz^2}\left(\frac{e^{-i\xi z}}{(z-i)^3}\right)_{z=-i}.$$

Now

$$\frac{d^2}{dz^2}((z-i)^{-3}e^{-i\xi z}) = \frac{d}{dz}(-i\xi(z-i)^{-3}e^{-i\xi z} - 3(z-i)^{-4}e^{-i\xi z})$$

$$= (-\xi^2(z-i)^{-3} + 6i\xi(z-i)^{-4} + 12(z-i)^{-5})e^{-i\xi z}$$

At $z = -i$ this is equal to

$$e^{-\xi}\left(\frac{-\xi^2}{8i} + \frac{6i\xi}{16} + \frac{12}{-32i}\right).$$

So

$$\int_{\gamma_R} \frac{dz}{(z^2+1)^3} = \frac{\pi e^{-\xi}}{4}(-\xi^2 - 3\xi - 3).$$

[10 marks]

Now

$$|z^2+1|^3 \geq (|z|^2-1)^3.$$

Also

$$|e^{-i\xi z}| \leq e^{\xi \mathrm{Im}(z)} \leq 1$$

if $\xi \geq 0$ and $\mathrm{Im}(z) \leq 0$. So

$$\left|\int_{\gamma_R'} \frac{e^{-i\xi z}dz}{(z^2+1)^3}\right| \leq \frac{\mathrm{Length}(\gamma_R')}{(R^2-1))^3} \to 0 \text{ as } R \to \infty.$$

[4 marks]

So

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{-i\xi x}dx}{(x^2+1)^3} = -\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{-i\xi z}dz}{(z^2+1)^3}$$

$$= \frac{\pi e^{-\xi}}{4}(\xi^2 + 3\xi + 3).$$

So for all ξ, using $\overline{\hat{f}(\xi)} = \hat{f}(-\xi)$ we have

$$\hat{f}(\xi) = \frac{\pi e^{-|\xi|}}{4}(\xi^2 + 3|\xi| + 3).$$

[2 marks].
2a)

$$\lim_{y\to 0}\left(\frac{1}{y} - \frac{1}{2\sin(\frac{1}{2}y)}\right) = \lim_{y\to 0}\frac{2\sin(\frac{1}{2}y) - y}{2y\sin(\frac{1}{2}y)}$$

$$= \lim_{y\to 0}\frac{2(\frac{1}{2}y - \frac{1}{3!}(\frac{1}{2}y)^3\cdots - y}{2y(\frac{1}{2}y - \frac{1}{3!}(\frac{1}{2}y)^3\cdots)} = \lim_{y\to 0}\frac{\frac{-2}{3!}(\frac{1}{2})^3 y\cdots}{1 - \frac{2}{3}y^2(\frac{1}{2})^3\cdots} = 0.$$

[An alternative method of doing this is to use l'Hopital's Rule - twice.] Hence, since $\sin(\frac{1}{2}y) \neq 0$ on $(0.\pi]$,

$$\frac{1}{y} - \frac{1}{2\sin(\frac{1}{2}y)}$$

extends to a continuous funciton on $[0,\pi]$, taking the value 0 at 0.
[5 marks]
b)

$$g(x-y) = \begin{matrix} 1 \text{ if } 0 \leq x-y \leq \pi, \text{ that is, } x-\pi \leq y \leq x \\ 0 \text{ if } -\pi < x-y < 0, \text{ that is, } x < y < x+\pi. \end{matrix}$$

So

$$S_n(g)(x) = \frac{1}{2\pi}\int_{x-\pi}^{x+\pi}\frac{\sin(n+\frac{1}{2})y}{\sin(\frac{1}{2}y)}dy$$

and because the integrand is even this is equal to

$$\frac{1}{2\pi}\left(\int_0^x + \int_0^{\pi-x}\right)\frac{\sin((n+\frac{1}{2})y)}{\sin(\frac{1}{2}y)}dy.$$

[3 marks]
By the Riemann Lebesgue Lemma, for all $a < b$,

$$\lim_{n\to\infty}\int_a^b \sin((n+(\frac{1}{2})y)\left(\frac{1}{2\sin(\frac{1}{2}y)} - \frac{1}{y}\right)dy = 0.$$

Applying this with $[a,b] = [0,x]$ and $[a,b] = [0,\pi - x]$, we have

$$\lim_{n\to\infty}\left(S_n(g)(x) - \frac{1}{\pi}\left(\int_0^x + \int_0^{\pi-x}\right)\frac{\sin((n+\frac{1}{2})y)}{y}dy\right) = 0.$$

[3 marks]
the Fourier Series Theorem says that

$$\lim_{n\to\infty} S_n(g)(x) = \frac{g(x+) + g(x-)}{2}.$$

2 marks

Put $x_n = \pi/(n+\frac{1}{2})$. Let $t = (n+\frac{1}{2})y$. Then $dt = (n+\frac{1}{2})dy$ and

$$\frac{dt}{t} = \frac{dy}{y}, \quad \sin((n+\frac{1}{2})y) = \sin t.$$

When $y = x_n$, $t = \pi$, and when $y = \pi - x_n$, $t = (n+\frac{1}{2})(\pi - x_n) \to 0$ as $n \to \infty$. So

$$\lim_{n\to\infty} S_n(g)(x_n) = \frac{1}{\pi}\left(\int_0^{\pi} + \int_0^{\infty}\right)\frac{\sin y}{y}dy = \frac{1}{2} + \frac{1}{\pi}\int_0^{\pi}\frac{\sin y}{y}dy.$$

[3 marks]

let

$$(-1)^n a_n = \int_{n\pi}^{(n+1)\pi}\frac{\sin y}{y}dy,$$

so that $a_n > 0$ for all n, $a_{n+1} < a_n$ and

$$\frac{\pi}{2} = \lim_{N\to\infty}\int_0^N \frac{\sin y}{y}dy = \sum_{n=0}^{\infty}(-1)^n a_n < a_1 = \int_0^{\pi}\frac{\sin y}{y}dy.$$

So

$$\lim_{n\to\infty} S_n(g)(x_n) = \frac{1}{2} + \frac{a_1}{\pi} > 1 = \frac{g(x_n+) + g(x_n-)}{2}.$$

[4 marks]

3a)

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

(One could use l'Hopital's Rule to get this since $\cos 0/1 = 1$, but it is a very standard limit. So $\sin x/x$, of whoch the denominator vanishes only at $x = 0$, extends to a continuous function on $[0,\pi]$ taking the vlaue 1 at 0, and is integrable on $[0,\pi]$. [3 marks]

However, if $x \in [n\pi + \frac{\pi}{4}, (n+1)\pi - \frac{\pi}{4}]$,

$$\left|\frac{\sin x}{x}\right| \geq \frac{1}{\sqrt{2}|x|} \geq \frac{1}{\sqrt{2}\pi(n+1)}.$$

So

$$\int_0^{\infty}\frac{|\sin x|}{|x|}dx \geq \sum_{n=0}^{\infty}\frac{1}{\sqrt{2}\pi(n+1)} = +\infty.$$

[4 marks]

3b) Now cosider the anticlockwise "beehive contour" $\gamma_{R,\epsilon}$ which is the union of $[-R,-\epsilon]$, $[\epsilon, R]$ and the semicircular arcs of radii R and ϵ in the upper half plane. let γ_R' and γ_ϵ' be the semicircular parts of the contour.

[2 marks]

The function $\frac{e^{iz}}{z}$ has no simgularities in or on $\gamma_{R,\epsilon}$. So by Cauchy's Theorem,

$$\int_{\gamma_{R,\epsilon}}\frac{e^{iz}}{z}dz = 0.$$

[1 mark] $|e^{iz}| = e^{-\mathrm{Im}(z)} \leq 1$ on γ_R'. in fact $|e^{iz}| \leq e^{-\sqrt{R}}$ if $\mathrm{Im}(z) \geq \sqrt{R}$. So, since the length of γ_R' is πR,

$$\left| \int_{\gamma_R'} \frac{e^{iz}}{z} dz \right| \leq \frac{\pi\sqrt{R}}{R} + \frac{R\pi e^{-\sqrt{R}}}{R} \to 0 \text{ as } R \to \infty$$

[It is acceptable to simply say something like: the integral along γ_R' tends to 0 as $R \to \infty$ by Jordan's Lemma.]
[2 marks]
So

$$\lim_{R\to\infty,\epsilon\to 0} \mathrm{Im} \left(\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \right) \frac{e^{ix}}{x} dx = \lim_{\epsilon\to 0} \mathrm{Im} \left(\int_{\gamma_\epsilon'} \frac{e^{iz}}{z} dz \right)$$

$$= \lim_{\epsilon\to 0} \mathrm{Im} \left(\int_0^{\pi} e^{i\epsilon e^{i\theta}} \frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}} d\theta \right) = \lim_{\epsilon\to 0} \mathrm{Im} \left(\int_0^{\pi} i(1 + i\epsilon e^{i\theta} + \cdots) d\theta \right) = \pi.$$

[5 marks]
So

$$\lim_{R\to\infty,\epsilon\to 0} \left(\int_{-R}^{-\epsilon} + \int_{\epsilon}^{R} \right) \frac{\sin x}{x} dx = \pi.$$

Since $\frac{\sin x}{x}$ is even,

$$\lim_{R\to\infty,\epsilon\to 0} \int_{\epsilon}^{R} \frac{\sin x}{x} dx = \lim_{R\to\infty} \int_0^R \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

[3 marks]
4(i) a) For all n we have

$$\hat{u}(1,n) = \hat{f}(n),$$
$$\hat{u}(R,n) = 0.$$

[1 mark]
(i)b) For $n = 0$ the differential equation becomes

$$r\frac{D^2}{dr^2}\hat{u}(r,0) + \frac{d}{dr}\hat{u}(r,0) = \frac{d}{dr}\left(r\frac{d}{dr}\right)\hat{u}(r,0) = 0.$$

So for a constant A,

$$r\frac{d}{dr}\hat{u}(r,0) = A,$$

and for a constant B,

$$\hat{u}(r,0) = A \ln R + B.$$

Then $\hat{u}(1,0) = \hat{f}(0)$ gives $B = \hat{f}(0)$ and $\hat{u}(R,0) = 0$ gives $A = -\hat{f}(0)/\ln R$. So

$$\hat{u}(r,0) = \hat{f}(0) - \hat{f}(0)\frac{\ln r}{\ln R}.$$

3 marks
(i)c) If $n \neq 0$, if we try a solution $\hat{u}(r,n) = r^{\lambda}$ we get

$$\lambda(\lambda-1)r^{\lambda-2} + \lambda r^{\lambda-2} - n^2 r^{\lambda-2} = 0.$$

So

$$\lambda^2 - n^2 = 0$$

and $\lambda = \pm n = \pm|n|$. So for constant A_n and B_n,

$$\hat{u}(r,n) = A_n r^{|n|} + B_n r^{-|n|}.$$

[2 marks]

Then $\hat{u}(R,n) = 0$ gives $A_n = -B_n R^{-2|n|}$ and $\hat{u}(1,n) = \hat{f}(n)$ gives

$$\hat{f}(n) = -B_n R^{-2|n|} + B_n$$

which gives

$$B_n = \frac{\hat{f}(n)}{1 - R^{-2|n|}}, \quad A_n = \frac{-\hat{f}(n)R^{-2|n|}}{1 - R^{-2|n|}}.$$

[2 marks]

So

$$|A_n \leq 2|\hat{f}(n)|R^{-2|n|} \text{ if } R \geq 2,$$

$$|A_n r^{|n|} \leq 2|\hat{f}(n)|R^{-|n|} \text{ for } 1 \leq r \leq R,$$

$$|B_n = \hat{f}(n)| \leq \frac{|f(n)|R^{-2|n|}}{1 - R^{-1}} \leq 2|\hat{f}(n)|R^{-2|n|} \text{ if } R \geq 2.$$

So

$$|\hat{u}(r,n) - r^{-|n|}\hat{f}(n)| \leq |\hat{f}(n)|R^{-|n|} \text{ if } 1 \leq r \leq R.$$

[3 marks]

(i)d)

$$1 + \sum_{n=1}^{\infty} r^{-|n|}(e^{in\theta} + e^{-in\theta}) = \sum_{n=0}^{\infty}(r^{-1}e^{i\theta})^n + \sum_{n=0}^{\infty}(r^{-1}e^{-i\theta})^n - 1$$

$$= \frac{1}{1 - r^{-1}e^{i\theta}} + \frac{1}{1 - r^{-1}e^{-i\theta}} - 1$$

$$= \frac{1 - r^{-1}e^{i\theta} + 1 - r^{-1}e^{-i\theta} - 1 + r^{-1}e^{-i\theta} + r^{-1}e^{i\theta} - r^{-2}}{(1 - r^{-1}e^{i\theta})\overline{(1 - r^{-1}e^{i\theta})}} = \frac{1 - r^{-2}}{|1 - r^{-1}e^{i\theta})}|^2$$

(ii) For any continuous 2π-periodic functions f and g on $\mathbf{R}$, if $h = f * g$,

$$\hat{h}(n) = \hat{f}(n)\hat{g}(n).$$

The function

$$g(\theta)1 + \sum_{n=1}^{\infty} r^{-|n|}(e^{in\theta} + e^{-in\theta})$$

has Fourier coefficients $2\pi r^{-|n|}$.

[1 mark]

So by the Fourier series Theorem,

$$\frac{\hat{f}(0)}{2\pi} + \sum_{n\neq 0} \frac{\hat{f}(n)r^{-|n|}}{2\pi}e^{in\theta} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1 - r^{-2}}{|1 - r^{-1}e^{it}|^2} f(\theta - t)dt,$$

$$u(r,\theta) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \hat{u}(r,n)e^{in\theta}.$$

[2 marks]

So by the estimate on $\hat{u}(r,n)$,

$$|u(r,\theta)+\frac{\ln r}{2\pi\ln R}\int_0^{2\pi}f(t)dt-\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^{-2}}{|1-r^{-1}e^{it}|^2}f(\theta-t)dt|$$

$$\leq\sum_{n\neq 0}4|hatf(n)|R^{-|n|}\leq 4\int_0^{2\pi}|f(t)|dt\sum_{n\neq 0}R^{-|n|}$$

$$\leq\frac{8R^{-1}}{1-R^{-1}}\int_0^{2\pi}|f(t)|dt\leq 16R^{-1}\int_0^{2\pi}|f(t)|dct$$

if $R\geq 2$.
[3 marks]
5.(i) *Tonelli's Theorem.* If one of

$$\int\int|f(x,y)|dxdy,\ \int\int|f(x,y)|dydx$$

is finite, then

$$x\mapsto\int f(x,y)dy,\ y\mapsto\int f(x,y)dx$$

are defined almost everywhere, and the double integrals

$$\int\int f(x,y)dxdy,\ \int\int f(x,y)dydx$$

are both defined and are equal.
[5 marks
(ii)

$$(\hat{f}\hat{g})^v(y)=\int_{-\infty}^{\infty}\hat{f}(\xi)\hat{g}(\xi)e^{i\xi y}d\xi=\int_{-\infty}^{\infty}\hat{g}(\xi)e^{i\xi y}\int_{-\infty}^{\infty}f(x)e^{-i\xi x}dxd\xi.$$

Now

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\hat{g}(\xi)||e^{i\xi(y-x)}||f(x)|dxd\xi=\int_{-\infty}^{\infty}|\hat{g}(\xi)|d\xi\int_{-\infty}^{\infty}|f(x)|dxd\xi<\infty.$$

So by Tonelli,

$$(\hat{f}\hat{g})^v(y)=\int_{-\infty}^{\infty}f(x)\int_{-\infty}^{\infty}\hat{g}(\xi)e^{i\xi(y-x)}d\xi dx=\int_{-\infty}^{\infty}f(x)(\hat{g})^v(y-x)dx$$

$$=\int_{-\infty}^{\infty}f(y-t)(\hat{g})^v(t)dt$$

(using the change of variable $t=y-x$, $dt=-dx$)

$$=f*(\hat{g})^v(y).$$

[5 marks]
(iii)

$$\int_{|y|\geq\delta}\frac{1}{\pi}\frac{\lambda}{\lambda^2+y^2}dy=\frac{2}{\pi}\lim_{\Delta\to\infty}[\arctan(y/\lambda)]_\delta^\Delta=\frac{\pi}{2}-\mathrm{Arctan}(y/\delta)\to 0\text{ as }\lambda\to 0.$$

$$\int_{-\infty}^{\infty}g_\lambda(y)dy=\lim_{\Delta\to\infty}\left[\frac{1}{\pi}\arctan(y/\lambda)\right]_{-\Delta}^{\Delta}=\frac{1}{\pi}\left(\frac{\pi}{2}-\frac{-\pi}{2}\right)=1.$$

[5 marks]

So

$$f(x) - \frac{1}{2\pi}(\hat{f}\hat{g}_\lambda)^v(x) = f(x) - f * g_\lambda(x) = \int_{-\infty}^{\infty} (f(x)g_\lambda(y) - f(x-y)g_\lambda(y))dy.$$

[1 mark]

Let $|f(y)| \le M$ for all y. Fix x. Given $\epsilon > 0$ choose $\delta > 0$ so that $|f(x) - f(x-y)| < \epsilon/2$ for all $|y| \le \delta$. Then for this δ, choose λ_0 so that

$$\int_{|y|\ge\delta} g_\lambda(y)dy \le \frac{\epsilon}{4M} \text{ for all } \lambda \le \lambda_0.$$

Then

$$\left|\int_{|y|\ge\delta} (f(x) - f(x-y))g_\lambda(y)dy\right| \le 2M \int_{|y|\ge\delta g_\lambda} (y)dy < \frac{\epsilon}{2} \text{ for all } \lambda \le \lambda_0.$$

Then

$$\left|f(x) - \frac{1}{2\pi}(\hat{f}\hat{g}_\lambda)^v(x)\right| \le \frac{\epsilon}{2}\int_{|y|\le\delta} g_\lambda(y)dy + \frac{\epsilon}{2} \le \epsilon \text{ for all } \lambda \le \lambda_0.$$

6a)

$$\hat{u}_{yy}(\xi, y) = \frac{\partial^2}{\partial y^2}\hat{u}(\xi, y),$$

$$\hat{u}_x(\xi, y) = i\xi\hat{u}(\xi, y),\ \hat{u}_{xx}(\xi, y) = -\xi^2\hat{u}(\xi, y).$$

So the transformed differential equation is

$$\frac{\partial^2}{\partial y^2}\hat{u}(\xi, y) - \xi^2\hat{u}(\xi, y) = 0,$$

and the transformed boudnary condition becomes

$$\hat{u}(\xi, 0) = \hat{f}(\xi).$$

[3 marks]

So

$$\hat{u}(\xi, y) = A(\xi)e^{|\xi|y} + B(\xi e^{-|\xi|y}.$$

[2 marks]

But

$$|\hat{u}(\xi, y)| \le \int_{-\infty}^{\infty} |e^{-i\xi x}u(x, y)dx \le M \text{ for all } \xi \in \mathbf{R},\ \text{all } y > 0.$$

So $A(\xi) = 0$,

[3 marks]

Then $B(\xi) = \hat{f}(\xi)$. So

$$\hat{u}(\xi, y) = \hat{f}(\xi)e^{-|\xi|y} = \hat{f}(\xi)\hat{g}_y(\xi)$$

where

$$g_y(t) = \frac{y}{\pi(y^2 + t^2)}.$$

So, since, for $h = f * g_y$, $\hat{h} = \hat{f}\hat{g}_y$ and h is deterkmined uniquely by its Fourier transform,

$$u(x, y) = f * g_y(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x-t)\frac{y}{y^2 + t^2}dt.$$

[3 marks]

6b) We have

$$\frac{u(x,y+h)-u(x,y)}{h}=\frac{1}{h}\int_0^h \frac{\partial u}{\partial y}(x,y+t)dt,$$

$$\frac{\partial u}{\partial y}(x,y)=\frac{1}{h}\int_0^h \frac{\partial u}{\partial y}(x,y)dt.$$

[2 marks]

So

$$\int_{-\infty}^{\infty} e^{-i\xi x}\frac{u(x,y+h)-u(x,y)}{h}dx-\int_{-\infty}^{\infty} e^{-i\xi x}\frac{\partial u}{\partial y}(x,y)dx$$

$$=\frac{1}{h}\int_{-\infty}^{\infty}\int_0^h e^{-i\xi x}\left(\frac{\partial u}{\partial y}(x,y+t)-\frac{\partial u}{\partial y}(x,y)\right)dtdx.$$

[2 marks]

But for $h\neq 0$,

$$\frac{1}{|h|}\int_{[0,h]}\int_{-\infty}^{\infty}\left|\frac{\partial u}{\partial y}(x,y+t)-\frac{\partial u}{\partial y}(x,y)\right) dxdt\leq \frac{2}{|h|}|h|\sup_y\int_{-\infty}^{\infty}\left|\frac{\partial u}{\partial y}(x,y)\right| dx<+\infty.$$

So by Tonelli we can cgange the order of integration.

[5 marks]

7(i)

$$\mathcal{L}(f)(z)=\int_0^{\infty} e^{-zt}f(t)dt.$$

[1 mark]

7(i)a) Put $g(t)=f(t-a)\chi_{[0,\infty)}(t-a)$. Then

$$\mathcal{L}(g)(z)=\int_0^{\infty} f(t-a)\chi_{[0,\infty)}(t-a)e^{-tz}dt=\int_a^{\infty} f(t-a)e^{-tz}dt$$

$$=\int_0^{\infty} f(u)e^{-(u+a)z}du=e^{-az}\mathcal{L}(f)(z),$$

using the change of varaiable $t-a=u$, $dt=du$, for which $u=0$ when $t=a$.

[2 marks]

7(i)b) We can find a sequence $\Delta_n\to+\infty$ such that $\lim_{n\to\infty}f'(\pm\Delta_n)=0$. Then by integration by parts,

$$\mathcal{L}(f')(z)=\int_0^{\infty} f'(t)e^{-tz}dt=\lim_{n\to\infty}\int_0^{\Delta_n} f'(t)e^{-tz}dt$$

$$=\lim_{n\to\infty}\left(\left[f(t)e^{-tz}\right]_0^{\Delta_n}+\int_0^{\Delta_n} zf(t)e^{-tz}dt\right)=-f(0)+z\mathcal{L}(f)(z).$$

[3 marks]

7(ii) Since $u(x,0)=\lim_{t\to 0}u_t(x,t)=0$, the transformed differential equation is

$$z^2\mathcal{L}(u)(x,z)=\frac{\partial^2}{\partial x^2}\mathcal{L}(y)(x,z)$$

with transformed boundary conditions

$$\mathcal{L}(u)(\ell,z)=0,\ \mathcal{L}(u)(0,z)=\mathcal{L}(g)(z).$$

So

$$\mathcal{L}(u)(x,z) = A(z)e^{zx} + B(z)e^{-zx},$$

$$A(z) + B(z) = \mathcal{L}(g)(z), \ \ A(z)e^{z\ell} + B(z)e^{-z\ell} = 0.$$

So

$$A(z) = -B(z)e^{-2z\ell} \text{ and } A(z) = \frac{-\mathcal{L}(g)(z)e^{-2z\ell}}{1 - e^{-2z\ell}}.$$

[5 marks]

So

$$\mathcal{L}(u)(x,z) = \frac{\mathcal{L}(g)(z)e^{-zx}}{1 - e^{-2z\ell}} - \frac{\mathcal{L}(g)(z)e^{(x-2\ell)z}}{1 - e^{-2z\ell}}$$

$$= \mathcal{L}(g)(z)\sum_{n=0}^{\infty} e^{-z(x+2n\ell} - \mathcal{L}(g)(z)\sum_{n=1}^{\infty} e^{z(x-2n\ell)}.$$

[3 marks]

So by (i)a)and the uniqueness of the function with a given Laplace transform,

$$u(x,t) = \sum_{n=0}^{\infty} g(t - x - 2n\ell)\chi_{[0,\infty)}(t - x - 2n\ell) - \sum_{n=1}^{\infty} g(t + x - 2n\ell)\chi_{[0,\infty)}(t + x - 2n\ell).$$

[3 marks]

For $t \leq \ell$,

$$\chi(t - x - 2n\ell = \begin{array}{l} 1 \text{ if } n = 0, t \geq x \\ = 0 \text{ otherwise,} \end{array}$$

$$\chi_{[0,\infty)}(t + x - 2n\ell) = 0 \text{ for all } n \geq 1.$$

So for $t \geq \ell$,

$$u(x,t) = \begin{array}{l} g(t - x) \text{ if } x \leq t, \\ = 0 \text{ if } t \leq x. \end{array}$$

[3 marks]

8a)

$$\hat{\mu}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi t} d\mu(t).$$

Fix ξ and $\epsilon > 0$. Choose R so that

$$\left| \left(\int_{-\infty}^{-R} + \int_{R}^{\infty} \right) d\mu(t) \right| < \frac{\epsilon}{3}.$$

Then choose $\delta > 0$ so that if $|\xi - \xi'| < \delta$ and $|t| \leq R$ then $|e^{-i\xi t} - e^{-i\xi' t}| < \epsilon/3$. Then for $|\xi - \xi'| < \delta$,

$$|\hat{\mu}(\xi) - \hat{\mu}(\xi')| \leq \int_{-\infty}^{\infty} |e^{-i\xi t} - e^{-i\xi' t}| d\mu(t)$$

$$\leq \int_{|t| \geq R} (|e^{-i\xi t}| + |e^{-i\xi' t}|) dt + \int_{|t| \leq R} |e^{-i\xi t} - e^{-i\xi' t}| d\mu(t) < \frac{2\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

[5 marks]

b) $*^n \mu = \nu_n$, $\hat{\nu}_n(\xi) = (\hat{\mu}(\xi))^n$.

[1 mark]

$$\hat{\mu}_n(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x/\sqrt{n}} d\nu_n = \hat{\nu}_n(\xi/\sqrt{n}) = \left(\hat{\mu}\left(\frac{\xi}{\sqrt{n}} \right) \right)^n.$$

[2 marks]
Central Limit Theorem Let μ be any probability measure with

$$\sigma = \int_{-\infty}^{\infty} x^2 d\mu(x) < +\infty,$$

and write

$$m = \int_{-\infty}^{\infty} x d\mu(x).$$

Then for any Borel-measurable $A \subset \mathbf{R}$,

$$\lim_{n\to\infty} \chi_A(x) d\mu_n(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_A e^{-(x-m)^2/2\sigma^2} dx.$$

[3 marks]
Let $\xi \geq 0$. Let γ_R denote the anticlockwise semicircular contour of radius R in the lower half plane, with straight line segment form R to $-R$ and semicircle arc γ_R'. We have

$$\hat{\mu}(\xi) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{1+x^2} dx.$$

We have $|e^{-i\xi z}| \leq 1$ for z$\in \gamma_R'$, $\xi \geq 0$, and $|z^2+1| \geq |z|^2 - 1$, and the length of γ_R' is πR. So

$$\left| \int_{\gamma_R'} \frac{e^{-i\xi z}}{1+z^2} dz \right| \leq \frac{\pi R}{R^2-1} \to 0 \text{ as } R \to \infty.$$

[3 marks] So

$$\begin{aligned} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-i\xi x}}{1+x^2} dx &= \lim_{R\to\infty} \frac{1}{\pi}\int_{-R}^{R} \frac{1}{\pi}\int_{-R}^{R} \frac{e^{-i\xi x}}{1+x^2} dx = -\lim_{R\to\infty} \int_{\gamma_R} \frac{e^{-i\xi z}}{1+z^2} dz \\ &= \frac{-2\pi i}{\pi \text{Res}\left(\frac{e^{-i\xi z}}{(z-i)(z+i),-i}\right) = e^{-\xi} Sosince} \end{aligned}$$

$\hat{\mu}(-\xi) = \overline{\hat{\mu}(\xi)}$, we have $\hat{\mu}(\xi) = e^{-|\xi|}$ for all $\xi \in \mathbf{R}$.
[3 marks]
We have

$$\hat{\nu}_n(\xi) = e^{-n|\xi|},$$

$$\hat{\mu}_n(\xi) = e^{-\sqrt{n}|\xi|} \not\to \frac{1}{\sqrt{2\pi}} e^{-\xi^2}.$$

The Central Limit Theorem fails because

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x^2}{1+x^2} dx = +\infty.$$