1. (a) The neighbourhood cats' chorus is made up of 5 ginger cats, 4 black cats, 2 grey cats and 3 tabby cats. Performances take place nightly between midnight and 5 a.m. A favourite item is a trio in which 3 cats take part at once.

(i) How many possible trios are there?

(ii) How many of these consist of 3 different-coloured cats?

(iii) Recently one of the cats was kept in at night. The remaining 13 cats gave a concert to cheer him up consisting of all possible trios, one after the other, lasting for one minute each.

Starting at midnight, were they able to finish this concert by their regular closing time?

(iv) As a grand finale the concert ended with 3 simultaneous quartets (groups of 4 cats) and a solo.

In how many ways could the selection of quartets for this finale be made?

[All answers should include a brief justification of the calculations].

(b) Prove that in any selection of 11 numbers from $\{1, 2, \ldots, 100\}$ there must be two numbers x, y with $0 < |\sqrt{x} - \sqrt{y}| < 1$.

(c) Let A be any selection of 9 numbers from $\{1, 2, \ldots, 30\}$. Prove that there must be two different (possibly overlapping) subsets of 3 numbers in A with the same sum.

[20 marks]

2. (i) Derive a formula for the number of ways of distributing r identical objects into n distinct containers.

Show why this formula also counts the number of solutions to the equation

$$k_1 + k_2 + \cdots + k_n = r$$

for non-negative integers $k_1, \ldots, k_n$.

(ii) Write down the coefficient of the monomial $x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n}$ in the expansion of $(x_1 + \cdots + x_n)^r$.

Write down also the number of different monomials which appear in this expansion.

(iii) The number of anagrams of the word MISSISSIPPI is the coefficient of one monomial in the expansion of $(M + I + S + P)^{11}$. Identify this monomial, and hence find the number of anagrams.

Find the number of anagrams in which SS does not occur.

(iv) State the inclusion-exclusion formula.

Find the number of integer solutions of $a + b + c + d = 28$, with

$$0 \le a \le 17,\ 0 \le b \le 13,\ 0 \le c \le 13,\ 0 \le d \le 11.$$

[20 marks]

3. State Hall's Assignment Theorem.

(a) A rectangular array of squares is given, from which a subset B of the squares is excluded.

(i) Show how the problem of constructing a perfect cover of the remaining squares by 2×1 tiles can be reformulated in terms of selecting distinct representatives for a suitable collection of sets, describing carefully what sets are to be used.

(ii) For each array below, where excluded squares are shown as ■, list a suitable collection of sets for this reformulation.

(iii) Using this collection of sets decide, with reasons, whether or not there is a perfect cover of the squares shown as □ by 2×1 tiles.

□ □ □ □ □ □	□ □ □ ■ □ □
□ □ □ □ ■ ■	□ ■ □ □ □ □
□ ■ ■ □ □ □	□ □ ■ ■ □ □
■ ■ □ □ ■ □	□ □ □ ■ □ □
■ □ □ □ □ □	□ □ ■ ■ □ ■

(b) A company has to allocate eight tasks $A, B, \ldots, H$ among seven staff. The staff are able to do certain of the tasks, as listed. Employee 1 can do A, B, H; 2 can do B, E, H; 3 can do C, D, F; 4 can do D, G; 5 can do A, E; 6 can do A, F, G; and 7 can do A, B, E.

The company initially allocates employees $1, \ldots, 6$ to tasks $A, \ldots, F$ respectively, but then realises that there is no task available for employee 7. Show how to reallocate the tasks so that all staff do one task, leaving one task unallocated, while employee 1 continues to do task A.

Employee 1 is subsequently absent for a period due to ill health, during which time the company employs a temporary substitute to do task A.

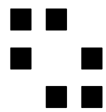
When employee 1 returns to work he is no longer fit to do task A, although he is still able to do B or H. Is it possible to reallocate the tasks among the original seven staff once the substitute has left, so that again all do one task, but employee 1 no longer does task A?

If the company decides instead to employ the substitute permanently on task A is it possible to reallocate all seven remaining tasks among the other seven staff?

[Justify your answers clearly.] [20 marks]

4. (i) Define a *rook polynomial.* Give rules which will enable the rook polynomial of any board to be calculated. State the 'forbidden positions' formula.

(ii) Calculate the rook polynomial of the 3×3 board shown.



(iii) The first row of a Latin square is

$$1\ 4\ 5\ 6\ 2\ 3\ .$$

Find how many possibilities there are for the second row.

(iv) The first two rows of a Latin square are

$$\begin{array}{c} 1\ 4\ 5\ 6\ 2\ 3 \\ 2\ 1\ 6\ 3\ 4\ 5\ . \end{array}$$

Find how many possibilities there are for the third row. [20 marks]

5. Solve the following recurrence relations. In each case you should find an expression for a_n and for the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$.

(i) $a_{n+2} = 4a_{n+1} - 4a_n,\ a_0 = 1,\ a_1 = 2.$

(ii) $a_{n+2} = 4a_{n+1} - 4a_n + 1,\ a_0 = a_1 = 1.$

(iii) $a_{n+1} = 3a_n + 2^{n+1},\ a_0 = 1.$

(iv) $a_{n+1} = 3a_n + 3^{n+1},\ a_0 = 1.$

(v) $a_{n+1} = 4a_n + 3(n+1),\ a_0 = 0.$

[20 marks]

6. The Catalan numbers $\{c_n\}$ are defined recursively by the relations

$$c_0 = 1, c_{n+1} = \sum_{r=0}^{n} c_r c_{n-r}, \ n \geq 0.$$

(i) Write down the sequence of these numbers up to c_5.

(ii) Prove that $c_n, \ n > 0$ counts the number of ways of cutting up a convex $(n+2)$-gon into n triangles by cuts from vertex to vertex.

(iii) Prove that c_n also counts the number of ways to pair $2n$ points on the circumference of a circle by n lines drawn across the circle which do not intersect each other.

(iv) Prove, by comparing the terms in x^{n+1}, that the generating function

$$C(x) = \sum_{n=0}^{\infty} c_n x^n$$

for the Catalan numbers satisfies the quadratic equation

$$C(x) - 1 = xC(x)^2.$$

Using this equation, show that $xC(x) = \frac{1}{2}(1 - (1-4x)^{\frac{1}{2}})$, and derive the formula

$$c_n = \frac{(2n)!}{n!(n+1)!}.$$

[20 marks]

7. Obtain a formula for the generating function $S(t) = \sum_{n=1}^{\infty} s_n t^n$, where s_n is the number of solutions of $n = a + 2b + 4c$ in non-negative integers.

Calculate $S(t)$ up to the term in t^9.

Establish an expression for the generating function $P(t)$ which enumerates the number of partitions of the positive integer n. Find also the function $P_m(t)$ which enumerates the number of partitions of n into parts of length $\leq m$.

Define the term *Ferrers graph.* Use Ferrers graphs to establish a bijection between partitions of n with at most m parts and partitions of n into parts of length $\leq m$.

Hence write down the generating function $R(t)$ which enumerates partitions of n with at most 4 parts.

Calculate $R(t)$ up to the term in t^9, and hence find the number of partitions of 9 with at most 4 parts.

Show that $(1 - t^4)R(t)$ enumerates partitions of n with at most 3 parts.

Use your calculations to determine the number of partitions of n with exactly 4 parts for all $n \leq 9$. Exhibit the corresponding Ferrers graphs in the case $n = 9$.

[20 marks]

8. Define the term *symmetric function.* For any positive integer n, define the *elementary symmetric function* σ_n and the *power sum symmetric function* π_n. State and prove the Newton Identities.

Express π_2 and π_3 in terms of the elementary symmetric functions and σ_2 and σ_3 in terms of the power sum functions.

Express each of the following

(a) in terms of the power sum functions π_1, π_2 and π_3 of α, β, γ, and

(b) in terms of their elementary symmetric functions σ_1, σ_2 and σ_3:

(i) $\alpha^3 + \beta^3 + \gamma^3$,

(ii) $\alpha^2(\beta + \gamma) + \beta^2(\gamma + \alpha) + \gamma^2(\alpha + \beta)$,

(iii) $\det \begin{pmatrix} 1 & 1 & 1 \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha^3 & \beta^3 & \gamma^3 \end{pmatrix} \Big/ \det \begin{pmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{pmatrix}$.

[20 marks]