1. (a) Define a *group*. Prove that each element of a group G has a unique inverse. Let a, b, c be elements in a group G. Find an expression for the element x satisfying the equation $axa^{-1}b = c$, and explain why x is an element of G.

(b) Let a, b be real numbers with $a \neq 0$. Define a map $f_{a,b} : \mathbf{R} \to \mathbf{R}$ by the rule

$$f_{a,b}(x) = ax + b$$

Obtain a formula which expresses the composite map $f_{a,b} \circ f_{c,d}$ in the form $f_{r,s}$ for suitably determined r, s. Deduce that the set, G, of all such maps

$$\{f_{a,b} : a, b \in \mathbf{R}, a \neq 0\}$$

is a non-abelian group under composition of functions. Show that $f_{-1,b}$ has order 2 (for all b). Are there any other elements of finite order?

2. State Lagrange's Theorem and use it to show that a group G with p elements (where p is a prime) is cyclic.

Now suppose that p is odd and let G be the dihedral group of symmetries of a regular p-sided polygon. Thus

$$G = \langle x, y : x^p = 1 = y^2, xy = yx^{-1} \rangle$$

where x corresponds to rotation through $360/p$ degrees and y corresponds to a reflection. (You may assume that G has $2p$ elements each of which is uniquely of the form $y^i x^j$ for $0 \leq i \leq 1$ and $0 \leq j \leq p-1$.)

Prove by induction on k that $x^k y = yx^{-k}$ and deduce that yx^k has order 2. Determine the complete list of the $p+3$ distinct subgroups of G. Show that every proper subgroup of G is cyclic, and explain why if H, K are distinct proper subgroups of G then $H \cap K = \{1\}$.

3 Let ϑ be a map between the groups $(G, \circ)$ and $(H, *)$. State what is meant by saying that ϑ is a homomorphism. Define the kernel and the image of ϑ, and state the *homomorphism* theorem.

Prove that the set of matrices of the form

$$A = \begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}$$

for $a, b \in \mathbf{Z}$ is a subgroup of the group of invertible 3×3 matrices with integer entries. Define a map f from G to the additive group of integers by $f(A) = a$ where A is as above. Prove that f is a homomorphism and deduce that G has a normal subgroup N with both N and G/N isomorphic to $\mathbf{Z}$. Show that G is an abelian group.

4 Let H be a subgroup of a group G of index 2. Show that H is a normal subgroup of G. Let H be the group of all even permutations on $\{1, 2, 3, 4\}$. List the elements of H together with their orders. Find a subset L of four elements of H which form a subgroup (prove that your subset *is* a subgroup). Find the list of distinct left cosets of $K = \langle(1\ 2\ 3)\rangle$ in H and also the list of distinct right cosets of K in H. Is K a normal subgroup of H?

Write down a formula which gives the number of elements in the set KL and deduce that $H = KL$. Show that H can be generated by an element π of order 3 together with two elements of order 2. By conjugating one of these elements of order 2 by π, prove that H can be generated by an element of order 2 and an element of order 3.

5 Let G be a finite group with n elements, say

$$G = \{g_1, g_2, \ldots, g_n\}.$$

For each element g of G define a map $\pi_g : G \to G$ by the rule $\pi_g(g_i) = g_j$ where $g_j = gg_i$. Prove that π_g is a permutation and that the map $g \mapsto \pi_g$ is an injective homomorphism from G into the group $S(n)$ of permutations on n symbols. Deduce that G is isomorphic to a subgroup of $S(n)$.

Explain why there are two isomorphism types of groups with four elements. Use the first paragraph to find subgroups of $S(4)$ isomorphic to each of the groups with 4 elements by renaming the four elements in each of these groups using the integers $\{1, 2, 3, 4\}$ and writing down the explicit permutations of the type π_g.

6 Define the terms *G-set*, *orbit* and *stabilizer* and state the orbit-stabilizer theorem. Given a subgroup H of a group G, define the normalizer $N_G(H)$ of H in G and prove directly that this is a subgroup of G. Show that H is a normal subgroup of $N_G(H)$.

Calculate $N_G(H)$ in each of the following cases:

(a) $G = D(4) = \langle x, y : x^4 = 1 = y^2, xy = yx^{-1}\rangle$ and $H = \langle x\rangle$;

(b) $G = D(4)$ and $H = \langle y\rangle$;

(c) $G = S(3)$ and $H = \{1, (1\ 2)\}$.

7 State the Sylow theorems. Prove that the number of Sylow p-subgroups of G is one if and only if this Sylow p-subgroup is a normal subgroup of G.
Establish the following claims:

(a) Let p and q be distinct prime numbers. Let G be a group of order pq with precisely one Sylow p-subgroup and precisely one Sylow q-subgroup. Then G is cyclic.

(b) Let G be a group with 12 elements which has more than one Sylow 3-subgroup. Then G has a unique Sylow 2-subgroup.

(c) A group with 66 elements has an element of order 33.

8 State the Jordan-Hölder Theorem explaining the terms you use.
(a) Give an example of a group with no composition series.
(b) Let G be a finite abelian group. Show that every chief series of G is a composition series.
(c) Show that $S(4)$ has a composition series which is not a chief series.

Define the term *simple group*. Prove that a simple abelian group is cyclic of prime order and give an example of a non-abelian simple group (without proof).