(January 2005)

Time allowed: Two Hours and a Half.

You may attempt as many problems as you like. The best FIVE answers will be taken into account. Each question carries the same weight.

1. [20 marks]

(i) Formulate and give the proof of the Cauchy–Riemann condition for a complex function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ to be holomorphic. Here x, $y \in \mathbb{R}$ and $u(x, y)$, $v(x, y) \in \mathbb{R}$ are the real and imaginary parts of $z \in \mathbb{C}$ and $f(z) \in \mathbb{C}$ respectively. [12 marks]

(ii) Apply the Cauchy–Riemann condition to check if the function $f(z) = f(x + iy) = (x^3 - 3xy^2) + i(3x^2y + y^3)$ (where $x, y \in \mathbb{R}$) is holomorphic.

[8 marks]

2. [20 marks]

(i) Find all zeros, poles and their orders of the meromorphic function

$$f(z) = \frac{(z^2 - 1)^2}{(z^2 + 4)^2\,(z - 2)}$$

on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. [7 marks]

(ii) Using the coordinate $\tau = \frac{1}{z}$ at ∞, find the first three terms $c_0+c_1\tau+c_2\tau^2$ of the power-series expansion of $f(z)$ at ∞. [8 marks]

(iii) Find the radius of convergence of the power series. [5 marks]

3. [20 marks]

(i) Find the residues of the meromorphic differential

$$\omega = \frac{z^3 + z + 1}{z^2} dz$$

at its poles on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. [12 marks]

(ii) Write down the sum of the residues. Is this predicted by the residue theorem? [2 marks]

(iii) Formulate the residue theorem for a compact Riemann surface. Sketch a proof of the theorem. [6 marks]

4. [20 marks]

Consider the holomorphic map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by the meromorphic function

$$f(z) = \frac{3z^2+1}{z^2-4}, \quad z \in \mathbb{C}.$$

(i) Find all ramification points and their orders for f. [12 marks]

(ii) Find the degree of f. [4 marks]

(iii) Check the statement of the Riemann-Hurwitz Formula for f. [4 marks]

5. [20 marks]

(i) Find the Riemann surface of the multi-valued holomorphic function

$$w = \sqrt{z+3} + \sqrt{z-5}, \quad z \in \mathbb{C}.$$

[15 marks]

(ii) Find the corresponding algebraic Riemann surface. [5 marks]

6. [20 marks]

Consider the complex curve X which is the projective closure of the curve

$$y^2 - x^3 + 2x^2 = 0$$

in $\mathbb{C}^2$.

(i) What are the singular points of X in $\mathbb{C}^2$? [5 marks]

(ii) Find a rational parametrisation $\overline{\mathbb{C}} \to X$ by the Riemann sphere.

[10 marks]

(iii) What corresponds to $\infty \in \overline{\mathbb{C}}$ for your parametrisation? [2 marks]

(iv) What corresponds to ∞ of X for your parametrisation? [3 marks]

7. [20 marks]

Consider the elliptic curve X given by the equation

$$\eta_0\eta_2^2 = 3\eta_1^3 - 2\eta_0^3$$

in projective coordinates $(\eta_0 : \eta_1 : \eta_2)$, or (equivalently) by the equation $y^2 = 3x^3 - 2$ in the affine coordinates $x = \frac{\eta_1}{\eta_0}$, $y = \frac{\eta_2}{\eta_0}$.

(i) Show that the meromorphic differential

$$\omega = \frac{dx}{y}$$

is holomorphic on X and has no zeros. [15 marks]

(ii) Give evidence why X is not isomorphic to the Riemann sphere $\overline{\mathbb{C}}$. [2 marks]

Show that X is a torus. [3 marks]

8. [20 marks]

(i) Formulate the Riemann-Roch Theorem for a compact Riemann surface of genus g. [5 marks]

Explain clearly what is meant by each term in the theorem. [5 marks]

(ii) Assume that X is a compact Riemann surface of genus 6 and A_1, A_2, A_3, A_4 are four distinct points of X.

Using the Riemann-Roch Theorem, find the dimension of the space of meromorphic functions on X which have poles of the orders not more than 3 at A_1, A_2, A_3, A_4 and which are holomorphic in all other points of X. [4 marks]

Prove that there exists a meromorphic function f on X which at the points A_1, A_2, A_3, A_4 has poles of orders exactly 3, and f is holomorphic in all other points of X. [6 marks]

9. [20 marks]

(i) What are canonical domains? [2 marks]

(ii) Explain their similarity and their difference. [5 marks]

(iii) Formulate the Riemann uniformisation theorem for Riemann surfaces. [4 marks]

(iv) Explain clearly what is meant by each term in the theorem. [5 marks]

(v) Give an example of the uniformisation of a Riemann surface which is different from the canonical domains. [4 marks]