THE UNIVERSITY of LIVERPOOL

JANUARY EXAMINATIONS 2007

NON-PHYSICAL APPLICATIONS II (POPULATION DYNAMICS)

TIME ALLOWED: Two Hours and a Half

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1. (a) The behaviour of a herd of deer on a remote Scottish island may be modelled by the Richards growth law

$$dN/dt = rN\left(1-(N/K)^3\right)$$

in which $N(t)$ is the population density of deer at time t and r and K are positive constants.

(i) State the biological significance of the constants r and K. (1 *mark*)

(ii) Sketch a graph of the function on the right-hand side of the above equation for $N \geq 0$. (2 *marks*)

(iii) Use your graph to show that the system possesses a single non-zero point equilibrium state (*Q*) and to predict the nature of the stability of this state. (2 *marks*)

(iv) Given that $N(0) = N_0$, use the substitution $u = N^{-3}$ to obtain the exact solution of the above equation. Show that this solution implies a behaviour which is consistent with your previous conclusion about the stability of the state *Q*. (5 *marks*)

(b) Suppose now that a discrete-time approach is taken to modelling the herd.

(i) Write down a difference equation for the discrete-time model analogous to the above, assuming that the time interval between successive generations of deer is τ. (2 *marks*)

(ii) Verify that this model predicts an equilibrium state *Q* with exactly the same population density as that obtained in (a). (1 *mark*)

(iii) Find the linearised difference equation describing the behaviour of the herd, according to this discrete model, in the neighbourhood of *Q*. (4 *marks*)

(iv) Deduce the range of values for which *Q* is locally stable. (2 *marks*)

(c) Compare and discuss briefly the conditions for the stability of *Q* as given by the analogous models in (a) and (b). (1 *mark*)

2. The dynamics of certain populations of insects is modelled by the recurrence relation

$$N_{t+1} = f(N_t) \quad \text{where} \quad f(N_t) = RN_t(1 - N_t)$$

where R>1 is a constant and N_t is the density at year t. (2 *marks*)

(i) Show that there is an equilibrium (Q) at $N^* = 1 - R^{-1}$.

(ii) Linearize about Q and hence find all values of R for which this equilibrium is stable. (4 *marks*)

(iii) Sketch the graph of f in each of two cases $R = 2$ and $R = 37/12$. On each graph sketch a ladder or cobweb diagram to show how an initial population near to zero evolves over time. (4 *marks*)

(iv) Find the explicit form of the two-step recurrence relation $N_{t+1} = f(f(N_t))$ for general R. Express the composite function as a polynomial and determine its coefficients.

(4 *marks*)

(v) In each of two cases $R = 2$ and $R = 37/12$ find all the feasible solutions of $N = f(f(N))$. (4 *marks*)

(vi) Comment briefly on what your results tell us about the behaviour of insect populations with $R = 2$ and $R = 37/12$. (2 *marks*)

3. The dynamics of a population of blowflies is modelled by the delay-time differential equation

$$\frac{dN}{dt} = \frac{bN}{\left(1 + mN(t-T)\right)} - dN$$

where *b*, *m* and *d* are positive constants.

(i) Discuss briefly the biological interpretations of *b*, *m* and *d*. (3 *marks*)

(ii) Find the value of $N^* \ (\neq 0)$ at equilibrium. Explain why this leads to the requirement $b > d$ and impose this on your further work. (3 *marks*)

(iii) Linearize about this equilibrium and hence find α such that the departure from equilibrium satisfies the equation

$$dx/dt = -\alpha x(t-T)\,.$$ (7 *marks*)

(iv) Try a solution of (ii) in the form $x(t) = x_0 \exp(\lambda t)$ and hence find a transcendental equation for λ involving α and T . (3 *marks*)

(v) Hence find the value of *T* at which a transition from stability to instability occurs. (4 *marks*)

4. Write down, in terms of the eigenvalues of the community matrix, the local stability conditions for, first, a continuous time model and, second, the corresponding discrete time model. (3 *marks*)

Let

$$F_1(N_1, N_2) = 6N_1(1 - \tfrac{1}{9}N_1 - \tfrac{8}{9}(N_1 + 1)^{-1}N_2)$$
$$F_2(N_1, N_2) = N_2(1 - N_1^{-1}N_2).$$

(i) Describe briefly the biological significance of the various terms in F_1 and F_2.

(3 *marks*)

(ii) Locate the coexistence equilibrium and investigate its stability in the linear approximation in the case where

$$dN_1/dt = F_1(N_1, N_2)$$
$$dN_2/dt = F_2(N_1, N_2).$$

(10 *marks*)

(iii) Determine the equivalent equilibrium and stability properties in the case where

$$\tau^{-1}[N_1(t+\tau) - N_1(t)] = F_1(N_1, N_2)$$
$$\tau^{-1}[N_2(t+\tau) - N_2(t)] = F_2(N_1, N_2),$$

in which τ (> 0) is a delay time. (4 *marks*)

5. State carefully a theorem involving a Liapunov function V which guarantees the asymptotic stability of an equilibrium point Q in population dynamics. (3 *marks*)

A 'toy' model for a predator-prey system is given by

$$dx/dt = -by - ax(9 - x^2 - y^2)(16 - x^2 - y^2)$$
$$dy/dt = \ \ bx - ay(9 - x^2 - y^2)(16 - x^2 - y^2)$$

where $x(t)$ and $y(t)$ denote departures from Q at time t and a and b are positive constants.

(i) Use the above theorem with $V(x, y) = x^2 + y^2$ to show that Q is an asymptotically stable equilibrium point. (4 *marks*)

(ii) Transform the above equations into polar coordinate form. Hence, using a graphical method, verify the result that Q is asymptotically stable. Show that the system possesses two other states of dynamic equilibrium and determine their stability. (8 *marks*)

(iii) Sketch trajectories in the *x*-*y* plane describing the behaviour of the system for $t \geq 0$ in the following cases:

$r_0 < 3, \quad r_0 = 3, \quad 3 < r_0 < 4, \quad 4 < r_0$ where $r_0 = \sqrt{{x_0}^2 + {y_0}^2}$ and $x_0 = x(0),\ y_0 = y(0)$.

(4 *marks*)

(iv) Explain briefly how the behaviour of the system would change if a became negative. (1 *mark*)

6. An ecosystem consists of four species with population densities at time t given by $N_1(t)$, $N_2(t)$, $N_3(t)$ and $N_4(t)$. The dynamics is modelled by the equations

$$\begin{aligned}
dN_1/dt &= N_1(-5 + N_2 + 3N_3 + N_4) \\
dN_2/dt &= N_2(-2 - N_1 + N_3 + 2N_4) \\
dN_3/dt &= N_3(3 - 3N_1 - N_2 + N_4) \\
dN_4/dt &= N_4(4 - N_1 - 2N_2 - N_3).
\end{aligned}$$

(i) Explain briefly the kind of interaction which occurs between each pair of species.

(3 *marks*)

(ii) Verify that the system possesses an equilibrium state (Q) with $N_1 = N_2 = N_3 = N_4 = 1$ and find the associated community matrix M. (5 *marks*)

(iii) Either by proving a general result about matrices with the same structure as M or otherwise show that the eigenvalues of M are purely imaginary. Comment briefly on the behaviour of the system close to Q. (6 *marks*)

(iv) Show that the quantity

$$\psi = N_1 + N_2 + N_3 + N_4 - \ln(N_1 N_2 N_3 N_4)$$

remains constant in time and state briefly how this is related to the nature of the stability of Q.

(6 *marks*)

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7. A number m of competing species occupy a habitat H and a single parameter θ describes the variation of a resource R in H.

(i) Explain <u>briefly</u> what is meant by the ecological niche for species i in H.

(2 *marks*)

(ii) Explain why the intensity of competition for R between species i and j can be measured by the quantity

$$\alpha_{ij} = \int_{-\infty}^{\infty} f_i(\theta) f_j(\theta) d\theta$$

in which $f_i(\theta)$ is the utilization function of species i. (4 *marks*)

(iii) Suppose that

$$f_i(\theta) = \left(C/\sqrt{w}\right)\exp\left(-(\theta - id)^2/(2w^2)\right),$$

where C is a constant. State how the values of the constants d and w are related to the intensity of the interspecific competition. Show that, for a suitable choice of C,

$$\alpha_{ij} = \alpha^{(i-j)^2} \quad \text{where} \quad \alpha_{ij} = \exp(-d^2/(4w^2)).$$ (7 *marks*)

(iv) Suppose that the population densities $N_i(t)$ of the species at time t are modelled by the Lotka-Volterra equations

$$dN_i/dt = N_i\left(k_i - \sum_{j=1}^{m} \alpha_{ij} N_j\right) \qquad i = 1,2,\ldots m$$

where the k_i are constants. Suppose further that the mismatch between resources available and consumed in H is given by

$$\Phi(t) = \Phi_0 + \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_{ij}(N_i - N_i^*)(N_j - N_j^*)$$

in which Φ_0 is a constant and the N_i^* $(\neq 0)$ are the population densities when the system is in equilibrium. Show that, as the system evolves in time, Φ decreases monotonically, towards a minimum value corresponding to equilibrium. (7 *marks*)