1. Two Russian security guards, Boris and Mikhail, are patrolling the grounds of the Irish Embassy in Moscow, when they stumble upon two unopened bottles of Guinness. Resisting the easy option of drinking one bottle each immediately, they agree to play a non-fatal version of Russian roulette to decide how their rare and warming find is to be disposed of on that cold and cheerless night.

Both Boris and Mikhail have several small bottles of cheap and nasty Vodka, which are used as stakes in the playing of the game. Mikhail's hat is large enough to contain the two bottles of Guinness and the bottles of vodka which the players wager during the game. Boris has a child's revolver which has a chamber with the capacity to hold four pretend bullets. If the trigger is pulled with a pretend bullet in line with the barrel, a loud but harmless explosion results. Otherwise a soft click is heard and the bullet chamber rotates through 90^o to the next position.

Rules of Russian Roulette

Each player wagers one bottle of vodka before the start of the game. Boris, being the senior guard, starts the game by placing one and only one pretend bullet in the revolver. He spins the chamber. He can either pull the trigger, wagering one bottle of vodka, or pass, wagering three bottles of vodka. If Boris decides to pull the trigger and an explosion is heard, Mikhail reclaims his hat and its contents. Otherwise, Mikhail takes the revolver. He has the same options and is required to make the same wagers as Boris, but does *not* spin the bullet chamber if he decides to pull the trigger. If Mikhail pulls the trigger and an explosion occurs, Boris takes the contents of Mikhail's hat. Otherwise the contents of the latter are shared equally.

Express this game in extensive form. Where chance events are involved, indicate the probabilities involved. Determine the possible outcomes from Boris's point of view. What is the simplest way of determining the corresponding outcomes from Mikhail's point of view?

Given that Boris has two pure strategies:

$$B_1 := (\text{Pull trigger});\ B_2 := (\text{Pass}),$$

write down Mikhail's pure strategies.

Calculate the first row of the strategic form of this game.

2. (i) Explain what is meant by a *Utility Function* in the context of lotteries over sure prospects. What condition is imposed on a utility if it satisfies the *Expected Utility Proposition*, EUP ?

A game player takes a view about a set of sure-prospects, regarding s as the least preferred and t as the most preferred. The player adopts a utility function U which satisfies the EUP. Show, using the standard type of transformation for EUP utilities, that there is an equivalent utility V, which satisfies $V(s) = 0$ and $V(t) = 1$. The utility V is said to be *Normalised.*

(ii) Abigail is presented with the possibility of taking part in various lotteries in which the prizes are amounts of money, in pounds sterling, between zero and ten. The following lotteries/sure-prospects are on offer :

$$\alpha : (50\% \text{ chance of winning } 5 \text{ and } 50\% \text{ chance of winning } 0\ );$$

$$\beta : (75\% \text{ chance of winning } 3.33 \text{ and } 25\% \text{ chance of winning } 0);$$

$$\lambda(p) : (100p\% \text{ chance of winning } 10 \text{ and } 100(1-p)\% \text{ chance of winning } 0);$$

$$s(x) : (\text{The sure-prospect of winning } x).$$

Abigail prefers α to $\lambda(0.25)$, and prefers $\lambda(0.25)$ to β.

Assuming the existence of a normalised EUP utility function V, draw the graph of $V(\lambda(p))$ as a function of p.

A second function F is defined on monetary prizes by $F(x) = 10V(s(x))$, for $0 \leq x \leq 10$.

Use Abigail's stated preference relations to find bounds for the values of $F(5)$ and $F(3.33)$. Hence indicate a possible shape of the function $F(x)$. Assuming $F(x)$ is continuous, deduce that at least three solutions of the equation $F(x) = x$ exist.

3. A zero-sum game has payoff matrix :

$$\begin{pmatrix} \alpha & 1 & 2 \\ -1 & 2 & -2 \\ -2 & \alpha - 3 & 3 \end{pmatrix}.$$

Determine the range of values of the parameter α for which a saddle point exists at the $(1,1)$ position. Find the unique value of α for which one player has a dominated strategy. In this case is the dominance strict or non-strict ?

Find the equilibrium strategy pair(s) in the case $\alpha = 2$.

4. (i) Why does the classic *Prisoners' Dilemma* game, in its noncooperative form, pose a dilemma for the players?

(ii) Many of the world's trading nations have negotiated multilateral reductions in their export subsidies through the General Agreement on Tariffs and Trade (GATT). However, the subsidy war described below, still rages on, and has yet to be resolved by the GATT.

It is well known that the soil conditions and climate in the neighbouring villages of Mickleshaw and Muckleshaw are ideal for growing giant leeks. These giant leeks do not travel well, so are only sold locally. Over many years, a healthy demand and fierce competitive spirit has developed for these leeks at the village markets. Furthermore, each village parish council has the legal right to subsidise its own farmers' cooperative at the level of £450 per month for the purpose of selling leeks at the other village's market.

If neither parish council pays a subsidy, then each cooperative makes a monthly profit of £625 on home sales and £400 on exports. If only one council decides to pay a subsidy, its local cooperative continues to make a monthly profit of £625 for home sales but an increased profit of £900 on exports. At the same time, their non-subsidised competitor faces a drop to £400 in monthly profits on home sales - the profit on exports is unaffected. If both councils pay the subsidy, each cooperative has monthly earnings of £400 on local sales and £900 on exports.

Considering each cooperative and its associated council as an economic unit, show that the situation described above can be modelled by a game of the Prisoners' Dilemma type.

Determine the maximin strategies for this game, and show that there are no mixed strategy Nash equilibria.

Explain the concepts of *A solution in the strict sense* and *A solution in the completely weak sense* with reference to this game.

Suggest a change in the law which would be beneficial to the economies of Mickleshaw and Muckleshaw.

5. A 2-player cooperative game has bimatrix :

$$\begin{pmatrix} (a,2) & (3,0) \\ (2,0) & (2,2) \end{pmatrix}.$$

Draw the attainment sets for this game in the cases $a = 2$, $2 < a < 3$ and $a = 3$. Also mark the Pareto optimal sets in these cases. Show that the maximin bargaining payoffs are

$$(\frac{3}{4} + \frac{3}{4}a, \frac{3}{2}),$$

if $2 < a < 3$. Determine the maximin bargaining payoffs in the cases $a = 2$ and $a = 3$. What is strange about your last result ?

Find the threat bargaining solution for $a = 2$.

6. (i) Explain what is meant by the *Core* and a *Stable Set* in the context of N-person games with a characteristic function.

(ii) A three-person game, with players A, B and C, has a characteristic function whose non-zero values are given by :

$$V(\{A\}) = 4;\ V(\{B,C\}) = 6;\ V(\{C,A\}) = 7;\ V(\{A,B\}) = 5;\ V(\{A,B,C\}) = 10.$$

Determine the set of imputations and the core of this game. Plot the core, preferably on a triangular coordinate diagram.

Also plot the set $\{(4, 6 - x, x) : 0 \leq x \leq 6)\}$ and show that it forms a stable set.

7. Two Greek farmers, Alpha and Delta, spend a day picking olives. By the end of the day, Alpha has picked 100 kilograms of green olives, and Delta has picked 100 kilograms of black olives. The next day they meet at the market to share the results of their labours.

The utility functions U_α and U_δ for these goods are

$$U_\alpha(a_g, a_b) = \frac{a_g a_b}{a_g + 4a_b} \text{ and } U_\delta(d_g, d_b) = \frac{d_g d_b}{9d_g + d_b},$$

for Alpha and Delta, respectively. Here, a_g and d_g are the amounts, in kilograms, of green olives that Alpha and Delta hold, respectively, and a_b, d_b are similarly defined.

Write down constraints on and relations between a_g, d_g, a_b and d_b.

Derive the equation

$$a_g a_b + 20a_g - 120a_b = 0,$$

for the contract curve.

The management of the market insists that trading in olives should be carried out by maximising the welfare function W, where

$$W = 49U_\alpha + 121U_\delta.$$

Show, without numerically determining the final trading position, that it must lie on the contract curve. Verify that $(a_g, a_b) = (60, 20)$ is a possible final trading position.

Assuming that the farmers' utilities are comparable, who has benefitted most from the day's trading ?