1. The following terms and distinctions are used in the theory of extensive games :

Game Tree; *State*; *Perfect/Imperfect Information*; *Nature*; *Information Set*; *Outcome*; *Pure Strategy*; *Strategic Form*.

Explain these terms with brief reference to a game or games of your own choice. You may, if you wish, refer to the game of Korean Nim, described below.

Construct the game tree, with payoffs computed, for Korean Nim. Why are the states of the game not in one-to-one correspondence with the configurations of armies of tin soldiers ? Enumerate the pure strategies of both players.

Korean Nim : This is a variant of the traditional Chinese game of Nim. Two players, Him and Kim, take it in turn to remove toy soldiers from two armies, containing, initially, two soldiers each. Additionally there are several soldiers stranded in no-man's-land.

Him is the first to play. At each turn, at least one soldier must be taken, but may only be removed from one army. No soldiers may be removed from no-man's-land except as described in the following paragraph.

The game ends when a player chooses to deplete the last non-empty army. If that involves the removal of N soldiers, his opponent then removes N soldiers from no-man's-land. Each player's payoff is the total number of soldiers that he has removed during the game.

2. A games player called Arthur views four sure prospects s_1, s_2, s_3 and s_4, in descending order of strict preference - s_1 is the most preferred. As a gambler, he also has a view about lotteries of the form

$$l(p) \sim [ps_1, (1-p)s_4],$$

for all $p \in [0, 1]$.

Arthur usually plays in a sober rational manner, employing a utility function which satisfies the *Expected Utility Proposition*, EUP. When he is a little merry, Arthur still plays rationally, but uses a non-EUP utility function. After a big loss, he plays rationally, but is risk-averse. Very occasionally, Arthur plays in a downright irrational way. State, giving reasons, which of the following putative utility functions, $U(l(p))$, correspond to Arthur's four modes of gambling :

$$5 - 40p + 60p^2,\ 5 + 20p,\ 5 + 5p + 15p^2,\ 5 + 30p - 10p^2.$$

On an occasion when Arthur is completely sober, he plays a zero-sum game against Bert, with the outcome matrix

$$\begin{pmatrix} s_1 & s_4 \\ s_3 & s_2 \end{pmatrix}.$$

In a one-off play of the game, Arthur and Bert both choose their maximin pure strategy. State the outcome.

Why is it not possible, with the information given above, for Arthur and Bert to calculate their optimal mixed strategies, if the game is to be played many times ? Overcome this difficulty by providing your own choice of the missing information.

3. (i) A spectator at a 2-player constant-sum game is told that the row player has payoff matrix :

$$\begin{pmatrix} 9 & 21 \\ 18 & 3 \end{pmatrix}.$$

and that when optimal mixed strategies are played, the column player receives an average payoff of 3 units less than the row player.

Determine the row player's optimal mixed strategy and expected payoff. Deduce the column player's payoff matrix. Transform the row player's payoff matrix to zero-sum form, in such a way as to preserve the difference of 3 units in the players' average payoff.

(ii) A zero-sum game has payoff matrix :

$$\begin{pmatrix} 4 & 3 & 8 \\ 9 & 5 & 1 \\ 2 & 7 & 6 \end{pmatrix}.$$

Show that in the optimal mixed strategy pair for this game, both players use all pure strategies with equal frequency.

4. (i) Find the maximin mixed strategies and corresponding values for the non-cooperative game with bimatrix :

$$\begin{pmatrix} (1,0) & (3,2) \\ (5,1) & (2,0) \end{pmatrix}.$$

Find, by inspection of the bimatrix, two pure strategy equilibria. Use the swastika method to determine the other, mixed strategy, equilibrium. For all equilibria, give the strategies and payoffs for both players.

(ii) Prove, for a general non-cooperative bimatrix game, that any pure strategy which is *Strictly Dominated* by another, is absent from any equilibrium strategy. Hence determine the equilibria for the game with bimatrix :

$$\begin{pmatrix} (6,6) & (2,0) & (3,4) \\ (2,0) & (5,2) & (4,3) \end{pmatrix},$$

and show that the game has a strict solution.

5. A 2-player cooperative game has bimatrix :

$$\begin{pmatrix} (5,1) & (7,4) & (1,10) \\ (1,1) & (9,-2) & (5,1) \end{pmatrix}.$$

Draw the attainment set for this game and mark the Pareto optimal set. Given that the maximin values for this game are 3 for the row player and 1 for the column player, indicate the negotiation set. Calculate the maximin bargaining payoffs.

The column player decides to discard his second strategy because he feels that it overly rewards the row player. The latter responds by proposing that the outcome to the game should be determined by threat bargaining. Which player loses most from this turn of events ?

6. (i) Explain what is meant by the term *Essential* in the context of N-person games with a characteristic function.

(ii) A three-person game, with players A, B and C, has a characteristic function whose non-zero values are given by :

$$V(\{B, C\}) = V(\{C, A\}) = V(\{A, B\}) = V(\{A, B, C\}) = 12.$$

Show that this game is essential, but has an empty core.

Use triangular coordinates to display the set of imputations which are dominated by the imputation $(5, 4, 3)$ with respect to the coalition $\{A, B\}$. Employing area as a measure, what fraction of the set of all imputations does this subset represent ?

(iii) A fourth player D joins the above game. The new characteristic function is an extension of that for the three-person game. The extra values are defined as follows :

(a) $V(\{D\}) = x$;

(b) D acts as a dummy whenever he is a member of a two-person coalition;

(c) $V(\{D\} \cup \mathcal{S}) = V(\mathcal{S}) + 2x$, for all two-person coalitions $\mathcal{S}$ of which D is not a member;

(d) $V(\{A, B, C, D\}) = 12 + y$.

Write down a relation between x and y which ensures that the function V is superadditive.

Show then that the core of this four-person game is empty if

$$y < 4 + 2x.$$

7. This question is about the *Edgeworth Box* treatment of market games with two traders engaged in the exchange of two commodties in the absence of money.

Explain the role of *Individual Rationality* and *Joint Rationality* in the transition from an initial endowment point to a final trading position. In the case when both traders use differentiable utility functions, write down equations defining the set of final trading positions - the *Pareto Optimal Set* and its subset, the *Contract Curve.*

(i) Two dealers, Nixon and Ford, trade in used cars at the Detroit automart. They have jointly cornered the market in Buicks and Pontiacs. Being smooth operators their preferences are represented by differentiable utility functions of the Cobb-Douglas type :

$$U_N(b_N, p_N) = b_N^{\alpha} p_N^{1-\alpha} \text{ and } U_F(b_F, p_F) = b_F^{\beta} p_F^{1-\beta},$$

where b and p denote numbers of Buicks and Pontiacs, respectively, subscripted with N or F according as they are held by Nixon or Ford. Determine the equation of the Pareto optimal set for such trading - you may assume that $0 < \alpha, \beta < 1$. If $\alpha = \beta$, verify that the Pareto optimal set is a straight line whose slope depends only on the ratio of the total number of Buicks to the total number of Pontiacs at the automart.

(ii) Also at the Detroit automart are Straight Alice and Bent Fender, who deal in Stock Cars and Dragsters. Their preferences are represented by the utility functions :

$$U_A(c_A, d_A) = c_A + d_A \text{ and } U_B(c_B, d_B) = \min(c_B, d_B),$$

where c and d denote numbers of Stock Cars and Dragsters, respectively, subscripted with A or B according as they are held by Alice or Bent. Initial holdings are: 10 Stock Cars and 6 Dragsters by Alice; 6 Stock Cars and 10 Dragsters by Bent.

Sketch a sample of their indifference curves, and indicate on your plot the Contract Curve and its extent.