**1.** The game of 'Threepenny Nim' for two players A and B is played with three coins. Initially one is 'tails' and two are 'heads'. Players make moves alternately, and a move consists of either turning over any coin which is 'tails' or removing any (positive) number of coins which are 'heads'. The loser is the player who takes the last coin.

The initial state, the top level in the extensive form (tree diagram) can therefore be written

$$\mathrm{THH}_{(0)}$$

and the second level, after A's first move, will be

$$\mathrm{HHH}_{(1)} \qquad \mathrm{TH}_{(2)} \qquad \mathrm{T}_{(3)},$$

where the subscripts are simply to label the states.

Complete the tree diagram and add a final row indicating the losing player for each branch. [8 marks]

Show that A has a total of 5 strategies: A1 and A2, which both start from $(0) \to (1)$, but also specify a second move depending on B's first move; A3 and A4, similarly starting from $(0) \to (2)$; and A5 which is completely specified by $(0) \to (3)$. [4 marks]

List all B's strategies, showing that each is of the form

$$(1) \to (?); \quad (2) \to (?)$$

[4 marks]

Suppose that the winner gains a point and the loser loses a point. Write down the outcome matrix (for player A). [4 marks]

**2.** (a) A player has prospects $s_1$, $s_2$, $s_3$, and $s_4$ with $s_1$ preferred to $s_4$. Suppose that

$$s_2 \sim [\tfrac{1}{4}s_1, \tfrac{3}{4}s_4], \qquad s_3 \sim [\tfrac{3}{11}s_1, \tfrac{8}{11}s_4].$$

What is the preference relation between $s_2$ and $s_3$? [1 marks]

The prospect $s$ is such that

$$s \sim [\tfrac{1}{5}s_2, \tfrac{4}{5}s_3].$$

Find $p$ such that

$$s \sim [ps_1, (1-p)s_4],$$

[3 marks]

and $q$ such that

$$s_3 \sim [qs_1, (1-q)s_2].$$

[4 marks]

Is there an $r$ such that

$$s_3 \sim [rs_2, (1-r)s_4]?$$

Explain your answer. [2 marks]

(b) State the **Expected Utility Proposition** (**EUP**). [3 marks]

The EUP may be assumed to hold in the following example.

Let $u(x)$ be a person's utility of winning £$x$; and the utility function is standardised by calling $u(0) = 0$ and $u(100) = 100$. If he is indifferent between winning £40 for certain or taking part in a gamble where he has equal chances of winning £100 or £0, what is $u(40)$? [3 marks]

Given that he is willing to pay at least £10 to take part in a lottery where there is a probability of $\frac{1}{4}$ of winning and being paid £50, otherwise there is no payment, what can you say about his utility for losing £10? [4 marks]

**3.** (a) Find the solution of the following zero-sum game.

$$\begin{pmatrix} 1 & -2 \\ 3 & -3 \\ -1 & 1 \\ -2 & 3 \end{pmatrix}.$$

[9 marks]

(b) For the following zero-sum game compute the value, the optimal strategy for the row player, and *two* optimal strategies for the column player.

$$\begin{pmatrix} -1 & 1 & \frac{1}{2} \\ 1 & -1 & -\frac{1}{2} \end{pmatrix}.$$

[11 marks]

**4.** Show that the row player A will never play the second strategy in the following zero-sum game.

$$\begin{pmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & -1 \\ -2 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}.$$

[1 marks]

By considering the column player B show that the game can be further simplified to that given by the matrix

$$\begin{pmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

[1 marks]

Solve this game. [17 marks]

State the solution of the original $4 \times 4$ game. [1 marks]

**5.** Consider the cooperative game given by the bimatrix

$$\begin{pmatrix} (2,-1) & (-2,0) & (1,1) \\ (-1,2) & (0,2) & (1,-2) \end{pmatrix}.$$

Sketch the attainment set of this game. [3 marks]

Find the Pareto Optimal set. [2 marks]

Calculate the maximin-maximin pair for this game and so find the arbitration set. [7 marks]

Find the maximin bargaining solution of the game. [8 marks]

**6.** A three-person game has the following characteristic function.

$$V(1) = 4, \quad V(2) = V(3) = 0,$$
$$V(1,2) = 5, \quad V(1,3) = 7, \quad V(2,3) = 6,$$
$$V(1,2,3) = 10.$$

State the conditions satisfied by the imputations. [3 marks]

State the additional core conditions and find the core. [7 marks]

State the conditions satisfied by a set $S$ of imputations in order that it should form a stable set. [3 marks]

Show that the set

$$\{(4, 6-x, x), 0 \leq x \leq 6\}$$

is a stable set. [7 marks]

**7.** Arthur's Auntie Ann always gives him handkerchiefs for his birthday, while Bert's Uncle Boris always gives him ties. Arthur and Bert meet at a stage when they have accumulated 80 handkerchiefs and 40 ties respectively and decide that an exchange of ties and handkerchiefs could leave them both happier. Arthur's preferences for handkerchiefs and ties can be represented by the utility function

$$U_A(h,t) = (h+15)(2t+5),$$

while Bert's utility function is

$$U_B(h',t') = (h'+10)(t'+25),$$

where $h$, $h'$ are the holdings of handkerchiefs, $t$, $t'$ of ties.

Draw the Edgeworth box. (There is no need to show any indifference curves.) [3 marks]

State the condition satisfied by the contract curve and show that its equation is

$$14t - 9h = 100.$$

[8 marks]

Show that, if the exchange is based on a fixed relative value of handkerchiefs and ties, then 9 ties should be exchanged for 14 handkerchiefs, and find the final distribution of handkerchiefs and ties (to the nearest whole number). [9 marks]