1. Two opposing armies are advancing on two posts. The first army, under the command of Colonel Blotto, consists of five regiments, the second, under the command of General Feinstein, consists of four regiments. The army sending the most regiments to a post captures the post and all of the regiments sent there by the other side, scoring one point for the captured post and one for each of the captured regiments. (Each point won by one army is a point lost to the opposing army. If the two armies send the same number of regiments to a post then no points are scored by either side for that post.)

For each commander, denote by $\{x, y\}$ the pure strategy in which x regiments are sent to post 1, y to post 2. Construct the (6×5) points matrix for Colonel Blotto. [8 marks]

Show that there is no (pure strategy) saddle point in this game. [4 marks]

Suppose, now, that Colonel Blotto can never decide which post is more important and always uses mixed strategies in which, for example, $\{4, 1\}$ and $\{1, 4\}$ have equal probability. Show that, in this case, there *are* saddle points. What is the expected outcome for Colonel Blotto? [4 marks]

Does it make any difference to the expected outcome if General Feinstein (only), or both commanders, play only the symmetric strategies described above? [4 marks]

2. (a) State the **Expected Utility Proposition** (**EUP**). [4 marks]

(b) Assuming that the EUP holds, suppose that $U(x)$ is a person's utility function for winning £x, where $U(0) = 0$ and $U(100) = 100$. If she is indifferent between winning £30 for certain or taking part in a game where she has a 40% chance of winning £100 (and a 60% chance of winning £0), what is $U(30)$? [4 marks]

If she is willing to pay £10 (at least) to take part in a game where she has a 30% chance of winning the game and being paid £40, otherwise she gets no payment, what can you say about her utility for losing £10? [4 marks]

(c) Suppose that z, y, x, and w are sure prospects and that, for a particular game player,

$$z \succ y \succ x \succ w.$$

In addition, the following indifference relation holds:

$$[\tfrac{1}{4}w, \tfrac{3}{4}y] \sim [\tfrac{3}{4}x, \tfrac{1}{4}z].$$

Show that the assignments:

$$U(w) = 0, U(x) = 1, U(y) = 2, U(z) = 3;$$
$$U'(w) = -15, U'(x) = -8, U'(y) = -1, U'(z) = 6$$

are consistent with U and U' being utilities satisfying the EUP. Find a formula relating U and U'. [8 marks]

3. A gladiator team consisting of a Woman, a Lion, and a Cat has to fight one consisting of a Man, a Dog, and a Mouse. Each side can put forward one champion to fight and the chance of a win for the team (Woman, Lion, Cat) for each possible combination is given by

	Man	Dog	Mouse
Woman	0.5	x	0.1
Lion	x	0.7	0.8
Cat	0.2	0.5	0.9

In each of the following cases find the frequencies with which each team should select each champion and find the overall probability of victory for the team (Woman, Lion, Cat).

(a) with $x = 0.6$ [5 marks]

(b) with $x = 0.4$ [15 marks]

4. (a) Find the maximin-maximin pair and all equilibrium pairs of the following non-cooperative game:

$$\begin{pmatrix} (2,-3) & (-1,3) \\ (0,1) & (1,-2) \end{pmatrix}.$$

[10 marks]

(b) Define a solution in the **Nash Sense** and determine whether or not the game in part (a) has such a solution. [3 marks]

(c) Sketch the attainment set of the **cooperative** game with the same bimatrix as in (a). [2 marks]

(d) Define the term **Pareto Optimal** and show that $(2,-3)$, $(-1,3)$, and $(0,1)$ are Pareto Optimal points of the *cooperative* game. Say why they are also Pareto Optimal points of the *non-cooperative* game (a). [5 marks]

5. Firm A can make either 200 colour printers a week or 100 colour and 100 black and white printers, while firm B can make either 50 colour printers or 100 black and white printers each week. There is a weekly demand of 200 colour and 100 black and white printers; colour printers sell at £2000 each, and black and white at £1000 each. It costs firm A £1500 to make each colour printer and £600 to make each black and white printer. It costs firm B £1600 to make each colour printer and £850 to make each black and white printer. If more printers are made than there is demand, the firms sell the same proportion of what they made. Printers not sold are worth nothing, but the firms still have to pay the cost of manufacture. Set this up as a 2×2 two-person non-zero-sum game, where both firms are trying to maximise their profit.

Find, to the nearest pound, the profit each firm can ensure itself without cooperation. Describe the negotiation set of 'reasonable' profits if the two firms cooperate. Then find which member of this set is the maximin bargaining solution. [20 marks]

6. On Wimbledon Common, Bungo, the Womble, has found a mattress which he values at £2. Wellington has not got one, but would like just one mattress, which he would value at £6. Orinoco, who likes his comforts, also has found a mattress which he thinks is worth £4, but he would not mind a second one which he would value at £3. Set up the buying and selling as a three-womble game, and find its characteristic function. [6 marks]

Describe the set of imputations (x_B, x_W, x_O). What is the subset of these which form the core of the game? [7 marks]

Show that the imputations $\{(2+x, 4-x, 4), 0 \leq x \leq 4\}$ form a stable set. [7 marks]

7. Algernon and Bertrand are marooned on two neighbouring islands. Algernon's island produces 72 coconuts per year while Bertrand's produces 104 Kg of dates. At low tide it is possible to wade between the islands so that they are able to trade, but otherwise they prefer to keep themselves to themselves. Algernon's utility function for coconuts (x) and dates (y Kg) is

$$U_A = \frac{xy}{x + 6y},$$

while Bertrand's (x' for coconuts, y' Kg for dates) is

$$U_B = \frac{x'y'}{6x' + y'}.$$

Find the dimensions of the Edgeworth box for this situation, indicate the shape of an indifference curve for each player, and find the equation of the contract curve. [10 marks]

Since coconuts and dates are ready at different times of the year, Algernon and Bertrand decide to use some smooth stones, signed by both, as money. If one stone represents 13 coconuts, show that it should also buy about 45 Kg of dates, and find each person's available consumption of coconuts and dates for the year. [10 marks]