1. A game for two players, A and B, is played with four cards marked with the numbers 3, 3, 3, and 4. These cards are placed in a hat and each player stakes himself in with a £1 note. Each player draws one card and looks at it without allowing his opponent to see it. They then simultaneously raise the stakes by £0 or £4. If a player's stake is higher than his opponent's, he wins and takes the complete kitty. If the stakes are equal, each player draws a further card; then the player with the larger sum on his two cards wins.

Draw a game-tree for this game, including probabilities, and show the outcomes for player A. Identify the information sets, and find the pure strategies for each player. [20 marks]

2. The game players Archie and Bernard have VNM utility functions U_A and U_B, respectively, where the values for the sure prospects s_i, $i = 1, 2, ..., 6$, are given in the table:

	s_1	s_2	s_3	s_4	s_5	s_6
U_A	1	4	-2	-5	2	4
U_B	3	1	5	7	1	-3

Let $s(p)$ denote the risky prospect $[ps_2, (1-p)s_4]$.

Plot the function $U_A(s(p))$ for $0 \leq p \leq 1$, marking the values of p for the risky prospects, with respect to s_2 and s_4, which are equivalent to each of the sure prospects s_i, $i = 1, 2, ..., 6$. Why should s_6 not appear on a plot of $U_B(s(p))$?

Let the payoff pairs $(U_A(s_i), U_B(s_i))$ be denoted by t_i, for $i = 1, 2, ...6$.

The bimatrices $\mathbf{M}_1$ and $\mathbf{M}_2$ are given by

$$\mathbf{M}_1 = \begin{pmatrix} t_1 & t_2 \\ t_3 & t_4 \end{pmatrix}, \quad \mathbf{M}_2 = \begin{pmatrix} t_1 & t_3 \\ t_5 & t_6 \end{pmatrix}.$$

Archie and Bernard play two games: Game 1 has payoff matrix $\mathbf{M}_1$; Game 2 is a cooperative game with payoff matrix $\mathbf{M}_2$.

Draw the attainment sets for Games 1 and 2.

Deduce that Game 1 is completely antagonistic. Transform Game 1 to an explicitly zero sum game and solve it.

Find the Pareto Optimal Set for Game 2. [20 marks]

3. Two croquet clubs, the Anfield Avengers and the Goodison Gorgons, each has three teams at its disposal. Every week throughout the summer, each club captain has to select a team to play the other in a Saturday afternoon match, without knowing which team the other club captain is likely to select. The probablities of a win by the Avengers, corresponding to the various possible pairings of opposing teams, known from past experience, are given by the following form matrix, in which the rows are labelled by teams from the Avengers club and the columns by teams from the Gorgons club:

$$\begin{pmatrix} 0.8 & 0.2 & 0.4 \\ 0.4 & 0.5 & 0.6 \\ 0.1 & 0.7 & 0.3 \end{pmatrix}.$$

Regarding this as the payoff matrix for the row player of a two-player game, show that the game is equivalent to a zero sum game.

Find the frequencies with which the club captains should select the various teams in order to obtain the greatest number of victories on average. Which is the more successful club? [20 marks]

4. (a) Find the solutions of the following zero-sum games with pay-off matrices

$$\text{(i)} \quad \begin{pmatrix} 4 & -1 & 2 & 3 \\ 4 & 6 & 3 & 7 \\ 1 & 2 & -2 & 4 \end{pmatrix},$$

$$\text{(ii)} \quad \begin{pmatrix} -1 & 2 \\ 1 & -2 \\ 0 & -1 \end{pmatrix},$$

$$\text{(iii)} \quad \begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 0 & 1 & 3 \end{pmatrix}.$$

[14 marks]

(b) Find the equilibrium points of the following noncooperative game and determine which are Pareto optimal.

$$\begin{pmatrix} (0,-3) & (1,0) \\ (1,1) & (0,0) \end{pmatrix}.$$

Does this game have a solution in the strict sense? [6 marks]

5. A 2-player cooperative game has bimatrix:

$$\begin{pmatrix} (5,1) & (7,4) & (1,10) \\ (1,1) & (9,-2) & (5,1) \end{pmatrix}.$$

Draw the attainment set for this game and mark the Pareto optimal set. Given that the maximin values for this game are 3 for the row player and 1 for the column player, indicate the negotiation set. Calculate the maximin bargaining payoffs.

The column player decides to discard his second strategy because he feels that it overly rewards the row player. The latter responds by proposing that the outcome to the game should be determined by threat bargaining. Which player loses most from this turn of events? [20 marks]

6. Explain the superadditivity property of the characteristic function of an n-person cooperative game. What is a **dummy** in such a game?

Five boys, Alec, Basil, Cecil, Dennis, and Eric, meet behind the scout hut to consider the exchange of champion conkers.

Alec and Basil each have such a conker, which they value at 4 gobstoppers and 6 gobstoppers, respectively. Cecil, Dennis, and Eric do not possess a conker, but each would like to acquire one (and only one) and would value one at 5 gobstoppers, 3 gobstoppers, and 7 gobstoppers, respectively.

Regard the meeting of the boys as a 5-person cooperative game with transferable utilities, and for which each boy's utility has zero contribution from an initial holding of gobstoppers.

Justify the statement that one boy is a dummy in this game.

Calculate the values of the characteristic function for the 4-person game without the dummy.

Write down the conditions for an imputation to be in the core.

Show that, in the core, Basil gets six gobstoppers and Cecil gets none.

Given that gobstoppers are indivisible, what does the core predict for the other players? [20 marks]

7. Abdul and Benjamin are traders in sheep and goats at the Shalom cattle market. On one particular day, Abdul comes to market with 50 sheep and 10 goats, and Benjamin comes with 30 sheep and 70 goats.

The traders' preferences for sheep and goats are represented by the utility functions:

$$U_A(s, g) = (s + 15)(2g + 10), \; U_B(s', g') = (s' + 10)(g' + 50),$$

where s, s' are the holdings in sheep, and g, g' are the holdings in goats.

Draw the Edgeworth box and sketch very roughly a few indifference curves for each trader.

Show that the Pareto optimal set for trading without money is the line $7g - 9s = 100$. Determine the end points of the contract curve. [20 marks]