1. In Russian roulette, Alex and Boris each put 1000 roubles in the 'pot'. Alex must then *either* put in another 1000 roubles *or* spin the cylinder of a six-shot gun which has one bullet in it and fire it at his own head. If he is able, he then passes the gun to Boris, who also must either add another 1000 rubles to the 'pot' or spin the cylinder and fire at his own head. If both are still alive after this, they share the 'pot' equally, otherwise the survivor takes all.

Set this game up in extensive form, i.e., draw the tree graph, and find the outcome in terms of roubles for Alex for each branch. [8 marks]

Describe all the strategies for each player. [4 marks]

Show that, if Alex passes and Boris spins, then the expected value of Alex's outcome is a *loss* of 250 roubles. Find Alex's expected outcome if Alex spins and Boris's strategy in that case is to spin too if Alex survives. [5 marks]

Assuming that only the monetary outcome is of interest to the players (!), which strategy will each choose? [3 marks]

2. (i) Explain what is meant by a *Utility Function* in the context of lotteries over sure prospects. What condition is imposed on a utility if it satisfies the *Expected Utility Proposition*, EUP ? [4 marks]

A game player takes a view about a set of sure-prospects, regarding s as the least preferred and t as the most preferred. The player adopts a utility function U which satisfies the EUP. Show, using the standard type of transformation for EUP utilities, that there is an equivalent utility V which satisfies $V(s) = 0$ and $V(t) = 1$. The utility V is said to be *Normalised.* [4 marks]

(ii) Abigail is presented with the possibility of taking part in various lotteries in which the prizes are amounts of money, in pounds sterling, between zero and ten. The following lotteries/sure-prospects are on offer:

$$\alpha : \text{(50\% chance of winning 5 and 50\% chance of winning 0);}$$

$$\beta : \text{(75\% chance of winning 3.33 and 25\% chance of winning 0);}$$

$$\lambda(p) : (100p\% \text{ chance of winning 10 and } 100(1-p)\% \text{ chance of winning 0});$$

$$s(x) : (\text{the sure-prospect of winning } x).$$

Abigail prefers α to $\lambda(0.25)$, and prefers $\lambda(0.25)$ to β.

Assuming the existence of a normalised EUP utility function V, draw the graph of $V(\lambda(p))$ as a function of p. [2 marks]

A second function F is defined on monetary prizes by $F(x) = 10V(s(x))$, for $0 \leq x \leq 10$.

Use Abigail's stated preference relations to find bounds for the values of $F(5)$ and $F(3.33)$. Hence indicate a possible shape of the function $F(x)$. Assuming that $F(x)$ is continuous, deduce that at least three solutions of the equation $F(x) = x$ exist. [10 marks]

3. Solve each of the following two-player zero-sum games.

(a)

$$\begin{pmatrix} -1 & 1 & -2 & 0 \\ 1 & -1 & 2 & -1 \end{pmatrix},$$

[10 marks]

(b)

$$\begin{pmatrix} -1 & 3 \\ 4 & -1 \\ -3 & 5 \\ 3 & 1 \end{pmatrix}.$$

[10 marks]

4. A zero-sum game has payoff matrix

$$\begin{pmatrix} \alpha & 1 & 2 \\ -1 & 2 & -2 \\ -2 & \alpha - 3 & 3 \end{pmatrix}.$$

Determine the range of values of the parameter α for which one player has a dominated strategy. [4 marks]

Find the equilibrium strategy pair(s) and the value of the game in the case $\alpha = 2$. [16 marks]

5. The payoff bimatrix for a two-person non-zero-sum game is

$$\begin{pmatrix} (3,8) & (4,4) \\ (2,0) & (0,6) \end{pmatrix}.$$

Find all the equilibrium pairs when it is considered as a non-cooperative game. [4 marks]

Show that the first (row) player A can ensure that she gets 3, while the second (column) player B can be sure of $\frac{24}{5}$. [4 marks]

Find the maximin bargaining solution and the threat bargaining payoffs when the game is considered as a cooperative game. [10 marks]

Does B have an incentive to cooperate with A? Give a brief explanation. [2 marks]

6. In the local park there are two ponds, a large one and a small one. Four ducks, called Arthur, Buck, Chuck, and Donald, live on or near them. Three-quarters of the bread crumbs are thrown into the large pond by the people who use the park, and only a quarter of them are thrown into the small pond. Each duck tries to choose one of the ponds to live on and they can form gangs to frighten other ducks off their pond. Arthur can frighten off Buck and also Chuck and Donald individually, and even Chuck and Donald together. All other gangs of two or three ducks can frighten off any individual duck on his own. Buck can frighten off Chuck, and also Donald, while Chuck can frighten off Donald. When gangs of two ducks meet, Arthur and Buck can frighten off Chuck and Donald, Arthur and Chuck frighten off Buck and Donald, but Buck and Chuck frighten off Arthur and Donald. Set this up as a four-player game, where the payoff is the proportion of bread crumbs each duck gets. Find the characteristic function, the set of imputations, and the core of the game. [10 marks]

Which, if any, of the following sets of imputations is a stable set?

$$S_1 = \{(\tfrac{1}{4}, x, \tfrac{3}{4} - x, 0), 0 \le x \le \tfrac{3}{4}\};$$
$$S_2 = \{(\tfrac{1}{2} - x, x, x, \tfrac{1}{2} - x), 0 \le x \le \tfrac{1}{4}\}.$$

[10 marks]

7. Adam and Eve had no money, and so went foraging for food in the Garden of Eden. Adam found six figs and Eve found 18 apples. They then met to bargain over the figs and apples.

Adam's preferences are expressed by the utility function

$$U_A(x, y) = \frac{xy}{(9x + y)},$$

where x and y are his holdings of figs and apples, respectively, while Eve's utility function, in a similar notation, is

$$U_E(x', y') = \frac{x'y'}{(x' + y')}.$$

Draw the Edgeworth box and sketch very roughly a few indifference curves for each person. [5 marks]

Show that the contract curve has equation

$$27x - xy - 3y = 0.$$

[8 marks]

If Adam and Eve decide to exchange figs and apples on the basis of a fixed ratio, show that that ratio should be 3 apples for each fig. Hence find the number of figs which Adam and Eve have after their bargaining session: (a) if they work to the nearest whole number, and (b) if they are willing to cut the fruit. [7 marks]