PAPER CODE NO.
MATH326

THE UNIVERSITY
of LIVERPOOL

SUMMER 2003 EXAMINATIONS

Bachelor of Science : Year 3
Master of Mathematics : Year 3
Master of Mathematics : Year 4
Master of Physics : Year 4

RELATIVITY

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be counted.

The following results may be used freely as required

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}),$$

$$R^{\mu}{}_{\nu\sigma\rho} = \Gamma^{\mu}_{\nu\rho,\sigma} - \Gamma^{\mu}_{\nu\sigma,\rho} + \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\nu\rho} - \Gamma^{\mu}_{\alpha\rho}\Gamma^{\alpha}_{\nu\sigma},$$

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu},$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R,$$

$$c = 2.998 \times 10^{8} \text{ ms}^{-1}.$$

1. State one of Einstein's principles of special relativity.

   The Lorentz transformation between inertial frames S and S', where S' is moving at constant velocity v along the x-axis of S, is

   $$t' = \gamma(v)\left(t - \frac{v}{c^2}x\right) , \quad x' = \gamma(v)(x - vt) \ .$$

   If S'' is another inertial frame moving relative to S' at constant velocity u along the x'-axis, determine the velocity at which S'' travels relative to S.

   Using the Lorentz transformation show that

   $$t' - \frac{x'}{c} = k(v)\left(t - \frac{x}{c}\right),$$

   where

   $$k(v) = \left(\frac{c+v}{c-v}\right)^{1/2} .$$

   Show also that

   $$t' + \frac{x'}{c} = \frac{1}{k(v)}\left(t + \frac{x}{c}\right).$$

   Compute $(t'^{\,2} - x'^{\,2}/c^2)$ in terms of $(t^2 - x^2/c^2)$. What does this imply about $(t^2 - x^2/c^2)$?

   A particle is at rest at the origin O. At time $t = 0$ it decays into two identical particles which travel in opposite directions along the x-axis with speed 2c/3 relative to O. What is the speed of one particle relative to the other? If one of the particles sends a light signal with frequency 1000Hz, what is the frequency of the signal received by the other particle?

   [**20 marks**]

2. Define the momentarily comoving reference frame (MCRF).

A particle is moving with velocity $\mathbf{u} = (u_x, u_y, u_z)$ with respect to an observer in an inertial frame S. According to an observer in an inertial frame S', moving at constant relative velocity v in the positive x direction relative to S, the velocity of the same particle is $\mathbf{u}' = (u'_x, u'_y, u'_z)$ where

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \ , \qquad u'_y = \frac{u_y}{\gamma(v)\left(1 - \frac{u_x v}{c^2}\right)} \ .$$

Given that the Lorentz transformation between the coordinates is

$$t' = \gamma(v)\left(t - \frac{vx}{c^2}\right) \ , \qquad x' = \gamma(v)(x - vt) \ ,$$

show that the x and y components of the particle's acceleration $\mathbf{a} = d\mathbf{u}/dt$ and $\mathbf{a}' = d\mathbf{u}'/dt'$ according to the respective observers are related by

$$a'_x = \frac{a_x}{\gamma^3(v)\left(1 - \frac{u_x v}{c^2}\right)^3}$$

and

$$a'_y = \frac{1}{\gamma^2(v)\left(1 - \frac{u_x v}{c^2}\right)^2}\left(a_y + \frac{v u_y a_x}{c^2\left(1 - \frac{u_x v}{c^2}\right)}\right) \ .$$

If S' corresponds to the MCRF of the particle, show that $a_x = a'_x/\gamma^3(v)$ and $a_y = a'_y/\gamma^2(v)$.

Observer A is in a rocket whose engine produces a steady acceleration of 10ms^{-2} in the positive x direction as recorded by A. A second observer B in an inertial frame observes A passing in the positive x direction with zero velocity in the y direction and records a change in A's velocity v in the positive x direction by $\Delta v = 0.02\text{ms}^{-1}$ over a time interval $\Delta t = 0.01\text{s}$. Estimate the velocity v of A relative to B at this time.

[20 marks]

3. A particle of rest mass $5m$ is stationary at the origin and is struck by a photon of energy E travelling in the positive x direction to produce two identical particles of rest mass $3m$. They move off at the same speed and with total angle ϕ between them (where $\phi = 0$ corresponds to both particles moving in the positive x direction). Using conservation of energy-momentum, show that

$$\cos\phi = \frac{2E^2}{(E - mc^2)\,(E + 11mc^2)} - 1\,.$$

Sketch $\cos^2\theta$ as a function of E for $E > mc^2$, where $\theta = \phi/2$. What is the minimum value of E for which the process can occur? Show that ϕ must be less than 112.9^{o}. To what energy E does this value of ϕ correspond?

[20 marks]

4. Consider Cartesian coordinates $x^\mu = (x, y)$ and plane polar coordinates $x^{\mu'} = (r, \theta)$ where $x = r\cos\theta$ and $y = r\sin\theta$. Compute the inverse transformation matrix

$$(\Lambda^\mu{}_{\mu'}) = \frac{\partial x^\mu}{\partial x^{\mu'}} \equiv \Lambda^{-1}$$

as a function of r and θ and show that $\Lambda = \left(\Lambda^{\mu'}{}_\mu\right)$ is

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} \equiv \Lambda \ .$$

Let V^μ, $T^\mu{}_\nu$ be respectively rank 1 and rank 2 tensors. Write down the transformation rule for each tensor between coordinate systems x^μ and $x^{\mu'}$ and the definition of their covariant derivatives with respect to x^σ.

A vector V^μ has Cartesian components given by

$$V^\mu = \begin{pmatrix} x - y \\ x + y \end{pmatrix} \ .$$

Compute $T^\mu{}_\nu = V^\mu{}_{,\nu}$ and show that

$$\left(\Lambda T \Lambda^{-1}\right)^{\mu'}{}_{\nu'} \equiv \Lambda^{\mu'}{}_\mu T^\mu{}_\nu \Lambda^\nu{}_{\nu'} = \begin{pmatrix} 1 & -r \\ \frac{1}{r} & 1 \end{pmatrix} \ .$$

Using the vector transformation rule show that the components of $V^{\mu'}$ in the primed coordinate system as a function of r and θ are given by

$$V^{\mu'} = \begin{pmatrix} r \\ 1 \end{pmatrix}$$

and hence determine $V^{\mu'}{}_{,\nu'}$.

Compute $V^{\mu'}{}_{;\nu'}$ explicitly. Comment on its relation, if any, to the value of $(\Lambda T \Lambda^{-1})$ calculated above.

[You may use the fact that the only non-zero Christoffel symbols for plane polar coordinates are $\Gamma^r_{\theta\theta} = -r$ and $\Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r}$.]

[20 marks]

5. Consider a two-dimensional surface with line element given by

$$ds^2 = d\theta^2 + \mathrm{cosec}^2\theta d\phi^2 \ .$$

Write down $g_{\mu\nu}$ and deduce its inverse $g^{\mu\nu}$. Compute the Christoffel symbols $\Gamma^{\mu}_{\nu\sigma}$ and show that the only non-zero components are given by

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = -\cot\theta \quad , \quad \Gamma^{\theta}_{\phi\phi} = \mathrm{cosec}^2\theta \ \cot\theta \ .$$

Using these expressions show that the value of the independent component of the Riemann tensor is given by $R^{\theta}{}_{\phi\theta\phi} = -(2\cot^2\theta + 1)\mathrm{cosec}^2\theta$.

From the symmetry properties of the Riemann tensor deduce the value of $R^{\phi}{}_{\theta\phi\theta}$. Compute the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R. With these quantities show that the Einstein tensor vanishes.

[**20 marks**]

6. If U^{α} is a tangent vector to a curve, write down the condition for the curve to be a geodesic in terms of the covariant derivative of U^{α}.

Show that if a particle moves along a geodesic in a spacetime with metric $g_{\mu\nu}$ which does not depend on the coordinate x^{σ}, then the particle has a constant momentum p_{σ}.

The metric for a Schwarzschild spacetime is given by

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 \ ,$$

where $c = 1$ and M is mass of the gravitational source. A particle of mass m moves in this spacetime along a worldline in the plane $\theta = \pi/2$. Deduce that p_t and p_{ϕ} are constants.

If $p_t = m\tilde{E}$, $p_{\theta} = 0$ and $p_{\phi} = -m\tilde{L}$, where $\tilde{E}$ and $\tilde{L}$ are constants, show that

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 + \frac{\tilde{L}^2}{r^2}\right)\left(1 - \frac{2M}{r}\right) \equiv \tilde{E}^2 - \tilde{V}^2(r) \ ,$$

where τ is the proper time.

In the case where $\tilde{L}^2 = 16M^2$, sketch the function $\tilde{V}^2(r)$ in the region $r > 2M$, including the radius and nature of all turning points, zeros and asymptotes. What is the upper limit on the value of $\tilde{E}$ for a particle to be in an ellipitical orbit? What is the corresponding lower limit on its radius?

[**20 marks**]

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7. Consider a photon moving in the equatorial plane of the Schwarzschild spacetime whose coordinates are $x^\mu = (t, r, \theta, \phi)$ where $c = 1$. The trajectory is governed by

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - V^2(r)$$

and $d\phi/d\lambda = L/r^2$, where M is the mass of the gravitational source, λ parameterises the worldline of the photon and

$$V^2(r) = \frac{L^2}{r^2}\left(1 - \frac{2M}{r}\right) .$$

Plot $V^2(r)$ in the region $r > 2M$, clearly labelling all stationary points and showing the asymptotic behaviour. Describe the nature of the three possible trajectories.

For $M \neq 0$ and $Mu \ll 1$, show that ϕ satisfies

$$\frac{d\phi}{dy} = (1 + 2My)\left(\frac{1}{b^2} - y^2\right)^{-1/2}$$

where $b = L/E$, $u = 1/r$, $y = u(1 - Mu)$ and all terms of order y^3 and higher are neglected relative to y^2.

Show, by differentiating the right side of

$$\phi = \phi_o + \sin^{-1}(by) + \frac{2M}{b} - 2M\left(\frac{1}{b^2} - y^2\right)^{1/2} ,$$

where ϕ_o is a constant, that it satisfies the differential equation for $\phi(y)$. Using this solution, show that light is deflected by the gravitational source through an angle $\Delta\phi = 4M/b$.

[**20 marks**]