2MA66 Summer 1998

Instructions to candidates.

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be counted.

The following results may be used freely as required

$$\Gamma^{\mu}_{\alpha\beta} \;=\; \frac{1}{2} g^{\mu\nu}(g_{\nu\alpha,\beta} \;+\; g_{\nu\beta,\alpha} \;-\; g_{\alpha\beta,\nu})$$

$$R^{\mu}{}_{\nu\sigma\rho} \;=\; \Gamma^{\mu}_{\nu\rho,\sigma} \;-\; \Gamma^{\mu}_{\nu\sigma,\rho} \;+\; \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\nu\rho} \;-\; \Gamma^{\mu}_{\alpha\rho}\Gamma^{\alpha}_{\nu\sigma}$$

$$R_{\mu\nu} \;=\; R^{\sigma}{}_{\mu\sigma\nu} \quad , \quad R \;=\; R^{\mu}{}_{\mu}$$

$$G_{\mu\nu} \;=\; R_{\mu\nu} \;-\; \frac{1}{2} g_{\mu\nu} R$$

$$c \;=\; 2.998 \times 10^{8} \text{ ms}^{-1}$$

1. An inertial frame S' moves at constant speed u in the x-direction relative to an inertial frame S. Write down the Lorentz transformation relating coordinates (ct', x') in S' to coordinates (ct, x) in S, carefully defining any variables. Define the rapidity $\chi(u)$ of this transformation.

A third inertial frame S'' moves at constant speed v relative to S' along the positive x-axis. Show that frame S'' moves at speed w relative to S where

$$w = \frac{u+v}{1+uv/c^2} \quad .$$

If $|u| < c$ and $|v| < c$ show that $-c < w < c$.

A particle is at rest at the origin O. At time $t = 0$ it decays into two identical particles which move in opposite directions along the x-axis with constant speed $3c/5$. What is the speed of one particle relative to the other?

If instead one particle moves at speed $24c/25$ relative to the other, what was the original decay speed in the frame of O?

2. A rocket begins from rest at $t = 0$ at the origin O of an inertial frame S and undergoes a uniform proper acceleration a in the positive x-direction. Show that after time t in S the rocket will be at

$$x = \frac{c^2}{a}\left[\left(1+\frac{a^2t^2}{c^2}\right)^{1/2} - 1\right] \quad .$$

Sketch the spacetime diagram in (x, ct) coordinates, clearly showing the asymptotic behaviour for large times. After what time in S can the rocket no longer receive light signals from a stationary observer at O?

When the rocket is at the point $x = 2c^2/3a$ in S it receives a light signal from an observer at O. Illustrate this event on the spacetime diagram. At what time in S was the signal sent?

[You may quote the formula $d(u\gamma(u))/dt$ for the proper acceleration.]

3. A stationary particle of mass $5m$ is struck by a photon of energy E to produce two identical particles of mass $4m$. They move off with the same speed at an angle θ to the incident photon direction and on opposite sides. By considering the conservation of energy-momentum show that

$$\cos^2\theta = \frac{E^2}{(E+13mc^2)(E-3mc^2)} \quad .$$

Sketch the graph of $\cos^2\theta$ versus $x = E/mc^2$ in the region $x > 3$. By considering the minimum and maximum values that $\cos^2\theta$ can take in this region, deduce that θ must be less than $38 \cdot 68^{\text{o}}$ and that E must be at least $3 \cdot 9mc^2$.

4. Consider Cartesian coordinates $x^\mu = (x, y)$ and plane polar coordinates $x^{\mu'} = (r, \theta)$. If they are related by

$$x = r\cos\theta \qquad y = r\sin\theta$$

compute the transformation matrix

$$\left(\Lambda^\mu{}_{\mu'}\right) = \frac{\partial x^\mu}{\partial x^{\mu'}} \equiv \Lambda^{-1}$$

and show that $\Lambda = \left(\Lambda^{\mu'}{}_\mu\right)$ is

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{1}{r}\sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} \equiv \Lambda \ .$$

If the quantities V^μ and $T^\mu{}_\nu$ are tensors, write down the general transformation rule between their components in the primed system and those of the unprimed system in terms of Λ.

A vector V^μ has Cartesian coordinates $(-y, x)$. Compute $T^\mu{}_\nu = V^\mu{}_{,\nu}$ and show that

$$\Lambda^{\mu'}{}_\mu T^\mu{}_\nu \Lambda^\nu{}_{\nu'} = \begin{pmatrix} 0 & -r \\ \frac{1}{r} & 0 \end{pmatrix} \ .$$

Using the vector transformation rule compute the components of $V^{\mu'}$ in the primed coordinate system as a function of r and θ and hence determine $V^{\mu'}{}_{,\nu'}$.

Write down the definition of the covariant derivative of a vector and use it to compute $V^{\mu'}{}_{;\nu'}$. Comment on its relation, if any, to the value of $(\Lambda T \Lambda^{-1})$ above.

[You may use the fact that the Christoffel symbols for plane polar coordinates are

$$\Gamma^r_{\theta\theta} = -r \quad , \quad \Gamma^\theta_{\theta r} = \Gamma^\theta_{r\theta} = \frac{1}{r}$$

where the remaining components are zero.]

5. The metric for a sphere of unit radius is given by

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \ .$$

Write down $g_{\mu\nu}$ and deduce its inverse $g^{\mu\nu}$. Compute the Christoffel symbols $\Gamma^\mu_{\nu\sigma}$ and show that the only non-zero components are given by

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta \ , \ \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta \ .$$

Hence using these expressions calculate the value of the independent component of the Riemann tensor, $R^\theta{}_{\phi\theta\phi}$.

Compute the Ricci tensor and show that it is proportional to $g_{\mu\nu}$. Write down the value of the constant of proportionality.

6. If U is a tangent vector to a curve write down the condition for the curve to be a geodesic in terms of the covariant derivative of U.

Show that if a particle moves along a geodesic in a spacetime with metric $g_{\mu\nu}$ which does not depend on the coordinate x^σ, then the particle has a constant momentum component p_σ.

The metric for a Schwarzschild spacetime is

$$(ds)^2 = \left(1 - \frac{a}{r}\right) c^2 (dt)^2 - \left(1 - \frac{a}{r}\right)^{-1} (dr)^2 - r^2 (d\theta)^2 - r^2 \sin^2\theta \, (d\phi)^2$$

where a is a constant. A particle of mass m moves freely in the equatorial plane $\theta = \pi/2$. Deduce that p_0 and p_ϕ are constants where $x^0 = ct$.

Hence, if $(ds)^2 = c^2 (d\tau)^2$, $p_0 = m\tilde{E}/c$, $p_\theta = 0$ and $p_\phi = -\, m\tilde{L}$, show that

$$c^2 \left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - c^4 \left(1 + \frac{\tilde{L}^2}{c^2 r^2}\right)\left(1 - \frac{a}{r}\right) \equiv \tilde{E}^2 - c^4 \tilde{V}^2(r)$$

where τ is the proper time.

In the case when $\tilde{L}^2 = 4a^2c^2$ sketch the function $\tilde{V}^2(r)$ in the region $r > a$ and deduce the radius of the stable circular orbit and its value of $\tilde{E}$.

7. The equations governing the motion of a planet orbiting in the equatorial plane of a Schwarzschild spacetime are

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \left(1 + \frac{\tilde{L}^2}{r^2}\right)\left(1 - \frac{a}{r}\right) \quad \text{and} \quad \left(\frac{d\phi}{d\tau}\right)^2 = \frac{\tilde{L}^2}{r^4}$$

where $c = 1$. Obtain $(dr/d\phi)^2$ as a function of r.

Set $u = 1/r$ and show that

$$\left(\frac{du}{d\phi}\right)^2 = \frac{(\tilde{E}^2 - 1)}{\tilde{L}^2} + \frac{au}{\tilde{L}^2} - u^2 + au^3 \ .$$

By substituting $u = y + a/(2\tilde{L}^2)$ and neglecting terms of order y^3 and higher, show that the differential equation becomes

$$\left(\frac{dy}{d\phi}\right)^2 = \text{constant} + \frac{3a^3}{4\tilde{L}^4} y + \left(\frac{3a^2}{2\tilde{L}^2} - 1\right) y^2$$

where the explicit form of the constant is NOT required.

Consider a function of the form $y = y_0 + A\cos(k\phi)$ where y_0, A and k are constants. By differentiating this function once, find a differential equation for y which has the same form as the above equation. Deduce that there is a solution of this form if

$$k^2 = 1 - \frac{3a^2}{2\tilde{L}^2} \ .$$