2MA65 January 1997

In this paper bold-face quantities like **r** represent three-dimensional vectors.

Full marks can be obtained for complete answers to FIVE questions. Only the best FIVE answers will be counted.

1. A particle of mass m moves on the x-axis in a potential V such that

$$V = 0 \qquad 0 \leq x \leq L$$
$$V = \infty \qquad x < 0 \quad \text{and} \quad x > L.$$

Find the normalised eigenfunctions of the Hamiltonian, and show that the energy eigenvalues are E_n where

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2} \qquad n = 1, 2, 3 \ldots$$

At a certain instant the particle has the following normalised wave function:

$$\psi(x) = A\left(\sqrt{2}\sin\frac{\pi x}{L} + 2\sqrt{2}\sin\frac{2\pi x}{L} + 2\sqrt{2}\sin\frac{3\pi x}{L}\right) \qquad (0 \leq x \leq L),$$

where A is a real, positive normalisation constant.

(i) Write an expression for $\psi(x)$ in terms of $\phi_n(x)$, the normalised eigenfunctions of the Hamiltonian. Calculate the normalisation constant A.

(ii) What are the possible results of a measurement of energy and with what probability would they occur?

(iii) Show that the expectation value of the energy is given by

$$< H >= \frac{53}{18}\frac{\hbar^2\pi^2}{mL^2}.$$

2. A beam of identical particles of mass m and energy $E > 0$ is incident along the x-axis from $x < 0$ on a potential step

$$V(x) = V_0 \qquad x \geq 0$$
$$V(x) = 0 \qquad x < 0$$

where V_0 is a constant. Suppose that $E > V_0$.

(i) Write down the current density for a beam of particles with wavefunction $\psi(x) = Ae^{ikx}$. For the potential step above, calculate the reflection and transmission coefficients R and T, defined as the ratios of the reflected and transmitted current densities to the incident current density.

(ii) Compute the sum $R + T$, and comment on the result.

(iii) Consider the case $V_0 = -V_1$, with V_1 positive. What happens to R and T in the limit $V_1 >> E$? Is this surprising from the classical point of view?

3. The Hamiltonian for a particle of mass m moving on the x-axis in a harmonic oscillator potential is

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2$$

where the frequency ω is a positive constant.

(i) Show that the expectation value of H for any normalisable state of non-zero norm is positive or zero.

(ii) Given that the normalised ground state wave function is

$$\psi_0 = Ae^{-\frac{1}{2}\alpha^2 x^2},$$

where $A^2 = \frac{\alpha}{\sqrt{\pi}}$, determine the constant α and the ground state energy.

(iii) A particle is in the ground state ψ_0 of a harmonic oscillator potential with frequency ω. What is the expectation value of H in this state? The frequency is suddenly changed to ω'. By writing the new Hamiltonian H' in terms of the original Hamiltonian H, show that the expectation value of H' in the state ψ_0 is given by

$$< H' >= \frac{\hbar(\omega^2 + \omega'^2)}{4\omega}.$$

$[\int_{-\infty}^{\infty} x^2 e^{-\alpha^2 x^2} dx = \frac{\sqrt{\pi}}{2\alpha^3}]$

4. The angular momentum operators L_1, L_2 and L_3 satisfy the commutation relations

$$[L_1, L_2] = i\hbar L_3 \qquad \text{and cyclic permutations,}$$

which imply

$$[\mathbf{L}^2, L_1] = [\mathbf{L}^2, L_2] = [\mathbf{L}^2, L_3] = 0$$

(where $\mathbf{L}^2 = L_1^2 + L_2^2 + L_3^2$).

Suppose that $|l, m>$ are the normalised eigenstates such that

$$L_3|l, m>= \hbar m|l, m>, \qquad \mathbf{L}^2|l, m>= \hbar^2 l(l+1)|l, m> .$$

(i) Defining $L_+ = L_1 + iL_2$ and $L_- = L_1 - iL_2$, show that

$$[L_3, L_+] = \hbar L_+, \quad [L_3, L_-] = -\hbar L_-.$$

Hence, using also the commutation relations for $\mathbf{L}^2$ above, deduce that

$$L_+|l, m>= N_{l,m}|l, m+1>$$

and

$$L_-|l, m>= M_{l,m}|l, m-1>,$$

where $N_{l,m}$ and $M_{l,m}$ are constants.

(ii) A particle is in the angular momentum eigenstate $|1, 1>$. By writing L_1 in terms of L_+ and L_-, compute the uncertainty ΔL_1 of L_1 in this state, defined by $(\Delta L_1)^2 =< L_1^2 > -(< L_1 >)^2$.

[You may assume that in (i), $N_{l,m}$ and $M_{l,m}$ are given by

$$N_{l,m} = \hbar\sqrt{l(l+1) - m^2 - m}, \qquad M_{l,m} = \hbar\sqrt{l(l+1) - m^2 + m}.]$$

5. The Pauli matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $S_\theta = \frac{1}{2}\hbar\sigma_\theta$, where

$$\sigma_\theta = \sigma_1 \cos\theta + \sigma_2 \sin\theta,$$

be the spin operator in the direction in the xy plane making an angle θ with the x-axis.

(i) Show that $\sigma_\theta = \begin{pmatrix} 0 & e^{-i\theta} \\ e^{i\theta} & 0 \end{pmatrix}$. Compute the eigenvalues and the normalised eigenvectors of σ_θ. What are the possible results of a measurement of S_θ?

(ii) The Hamiltonian for a stationary electron of mass m and charge e in a magnetic field B along the z-axis is given by $H = \hbar\omega\sigma_3$, where $\omega = \frac{eB}{2m}$. By solving Schrödinger's equation, show that at time t the state of the electron is given by

$$\psi(t) = \begin{pmatrix} c_1 e^{-i\omega t} \\ \\ c_2 e^{i\omega t} \end{pmatrix},$$

where c_1, c_2 are constants.

(iii) Suppose that at time $t = 0$ the wave-function has the normalised form $\psi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$. Show that the expectation value of S_θ at time t is given by

$$< S_\theta >= \frac{1}{2}\hbar \sin(\theta - 2\omega t).$$

6. A particle of mass m moves in three dimensions under the influence of a Coulomb potential $V = -\frac{A}{r}$, where $r = |\mathbf{r}| = (x^2+y^2+z^2)^{\frac{1}{2}}$ and A is a positive constant.

(i) Given that the normalised ground state wave function is

$$\psi(\mathbf{r}) = Be^{-\frac{r}{a_0}}$$

where B and a_0 are constants, determine a_0, the ground state energy E_0, and B in terms of m, A and $\hbar$.

(ii) The particle is now subjected to an additional potential $\lambda r^2 \sin\theta$, where λ is a small parameter, and $\{r, \theta, \phi\}$ are spherical polar co-ordinates. Calculate the new ground state energy to first order in λ.

[You may assume that the radial part of the Laplacian in spherical polars is

$$\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r},$$

and also that

$$\int_0^\infty r^n e^{-\beta r} dr = \frac{n!}{\beta^{n+1}} \qquad (\beta > 0).]$$

7. State briefly how the variational method is used to estimate the ground state energy of a quantum mechanical system.

A particle of mass m moves on the x-axis subject to a potential $V(x) = \lambda |x|$, where λ is a positive constant. Using a normalised trial wave function $\psi(x) = Ae^{-\frac{1}{2}\beta^2 x^2}$, where $A^2 = \frac{\beta}{\sqrt{\pi}}$, show that

$$< H >= \frac{1}{4}\frac{\hbar^2 \beta^2}{m} + \frac{\lambda}{\sqrt{\pi}\beta}.$$

Hence use the variational principle to show that the ground state energy is given approximately by

$$E_0 = \frac{3}{4}\left(\frac{4\hbar^2\lambda^2}{m\pi}\right)^{\frac{1}{3}}.$$

[$\int_{-\infty}^{\infty} e^{-\beta^2 x^2} dx = \frac{\sqrt{\pi}}{\beta}$, $\qquad \int_{-\infty}^{\infty} x^2 e^{-\beta^2 x^2} dx = \frac{\sqrt{\pi}}{2\beta^3}$]