MATH323 FURTHER METHODS OF APPLIED MATHEMATICS
JANUARY 2007

1. Indicial equation

$$m^2 - 6m + 8 = 0 \Rightarrow (m-2)(m-4) = 0 \Rightarrow m = 2, 4.$$

So

$$y_{\mathrm{CF}} = c_1 e^{2x} + c_2 e^{4x}.$$

Try

$$\begin{aligned} y =& L_1(x)e^{2x} + L_2(x)e^{4x} \\ \Rightarrow L_1' e^{2x} + L_2' e^{4x} =& 0 \\ 2L_1' e^{2x} + 4L_2' e^{4x} =& \frac{e^{6x}}{1+e^{2x}} \\ \Rightarrow L_2' = -L_1 e^{-2x} \Rightarrow& L_1' = -\tfrac{1}{2}\frac{e^{4x}}{1+e^{2x}} \\ \Rightarrow L_1 = -\tfrac{1}{2}\int \frac{e^{4x}}{1+e^{2x}}dx \\ \text{Put } u = e^{2x} \Rightarrow& du = 2e^{2x}dx \Rightarrow L_1 = -\tfrac{1}{4}\int \frac{u}{1+u}du \\ = -\tfrac{1}{4}[u - \ln(1+u)] + c_1 =& -\tfrac{1}{4}[e^{2x} - \ln(1+e^{2x})] + c_1 \\ L_2' = \tfrac{1}{2}\frac{e^{2x}}{1+e^{2x}} \Rightarrow L_2 =& \tfrac{1}{4}\int \frac{1}{1+u}du \\ =& \tfrac{1}{4}\ln(1+e^{2x}) + c_2. \end{aligned}$$

So the general solution is

$$y = \{-\tfrac{1}{4}[e^{2x} - \ln(1+e^{2x})] + c_1\}e^{2x} + \{\tfrac{1}{4}\ln(1+e^{2x}) + c_2\}e^{4x}.$$

Now

$$\begin{aligned} y(0) = \tfrac{1}{2}\ln 2 \Rightarrow -\tfrac{1}{4} + c_1 + c_2 =& \tfrac{1}{2}\ln 2 \\ y(\tfrac{1}{2}\ln 2) = \tfrac{3}{2}\ln 3 \Rightarrow \left[-\tfrac{1}{4}(2 - \ln 3) + c_1\right]2 + \left[\tfrac{1}{4}\ln 3 + c_2\right]4 =& \tfrac{3}{2}\ln 2 \\ \Rightarrow 2c_1 + 4c_2 =& 1 \\ \Rightarrow c_1 = 0, \quad c_2 = \tfrac{1}{4} \\ \Rightarrow y = \tfrac{1}{4}\ln(1+e^{2x})(e^{2x} + e^{4x}). \end{aligned}$$

2. The Euler-Lagrange equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0.$$

If F does not depend explicitly on x then

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow \frac{\partial F}{\partial y'} = \text{constant} = C.$$

Now

$$\begin{aligned}
x^2 + x^2 y' = C \Rightarrow \frac{dy}{dx} &= \frac{C}{x^2} - 1 \\
\Rightarrow y &= -\frac{C}{x} - x + A
\end{aligned}$$

$$y(1) = 0 \Rightarrow -C - 1 + A = 0, \quad y(2) = -\tfrac{3}{2} \Rightarrow -\frac{C}{2} - 2 + A = -\tfrac{3}{2}$$

$$\Rightarrow A = 0, \quad C = -1.$$

$$\Rightarrow y = \frac{1}{x} - x, \quad y' = -\frac{1}{x^2} - 1$$

$$\begin{aligned}
\Rightarrow I &= -\int_1^2 x^2\left(\frac{1}{x^2} + 1\right)\left(1 - \frac{1}{2x^2} - \tfrac{1}{2}\right) dx \\
&= -\tfrac{1}{2}\int_1^2 (1 + x^2)\left(1 - \frac{1}{x^2}\right) dx \\
&= -\tfrac{1}{2}\int_1^2 \left(x^2 - \frac{1}{x^2}\right) dx \\
&= -\tfrac{1}{2}\left[\frac{1}{x} + \tfrac{1}{3}x^3\right]_1^2 \\
&= -\tfrac{1}{2}\left[-\tfrac{1}{2} + \tfrac{7}{3}\right] = -\tfrac{11}{12}.
\end{aligned}$$

The straight line joining $(1,0)$ to $(2,-\frac{3}{2})$ is $y = -\frac{3}{2}(x-1)$. So then

$$\begin{aligned}
I[y] &= \int_1^2 x^2\left(-\tfrac{3}{2}\right)\left(1 - \tfrac{3}{4}\right) dx = -\tfrac{3}{8}\int_1^2 x^2 dx \\
&= -\tfrac{3}{8}\left[\tfrac{1}{3}x^3\right]_1^2 = -\tfrac{7}{8}.
\end{aligned}$$

Now $-\frac{11}{12} < -\frac{7}{8}$ so the extremum is a minimum.

If $a = -1$ and $b = 2$ there is no extremal solution since there is a singularity in y at $x = 0$.

3. Define a new functional $I + \lambda J$ where λ is a Lagrange multiplier then apply the Euler-Lagrange equation to $F + \lambda G$. This determines y up to two arbitrary constants and λ which are found by the endpoint conditions and the constraint.

E-L gives

$$\begin{aligned} \frac{d}{dx}[2x^4 y'] + 4x^2 y =& \lambda x^2 \\ \Rightarrow x^4 y'' + 4x^3 y' + 2x^2 y =& \tfrac{1}{2}\lambda x^2 \\ \Rightarrow x^2 y'' + 4xy' + 2y =& \tfrac{1}{2}\lambda \end{aligned}$$

For the homogeneous equation try $y = x^n$, then

$$\begin{aligned} n(n-1) + 4n + 2 = 0 \Rightarrow& n^2 + 3n + 2 = 0 \\ \Rightarrow (n+1)(n+2) = 0 \Rightarrow& n = -1, n = -2 \end{aligned}$$

So (with the obvious particular integral of $y = \frac{1}{4}\lambda$,

$$y = \frac{A}{x} + \frac{B}{x^2} + \tfrac{1}{4}\lambda.$$

We require

$$\begin{aligned} \int_1^2 x^2 \left(\frac{A}{x} + \frac{B}{x^2} + \tfrac{1}{4}\lambda \right) dx =& 2 \\ \Rightarrow \left[\tfrac{1}{2}x^2 A + Bx + \tfrac{1}{12}\lambda x^3 \right]_1^2 =& 2 \\ \Rightarrow \tfrac{3}{2}A + B + \tfrac{7}{12}\lambda =& 2 \\ \text{Also } A + B + \tfrac{1}{4}\lambda =& -3, \\ \tfrac{1}{2}A + \tfrac{1}{4}B + \tfrac{1}{4}\lambda =& 2 \\ \Rightarrow A = 2, \quad B = -8, \quad \lambda = 12.& \\ \text{so } y =& \frac{2}{x} - \frac{8}{x^2} + 3. \end{aligned}$$

is the extremal curve.

4. The equations are of form

$$A\mathbf{u}_x + B\mathbf{u}_y = \mathbf{c}$$

where $\mathbf{u}_x = \begin{pmatrix} u_x \\ v_x \end{pmatrix}$, $\mathbf{u}_x = \begin{pmatrix} u_x \\ v_x \end{pmatrix}$, and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -3x^2 \\ -3x^2 & 0 \end{pmatrix}$$

$$\begin{aligned}
\Rightarrow \det(\lambda A - B) = \det \begin{pmatrix} \lambda & 3x^2 \\ 3x^2 & \lambda \end{pmatrix} &= 0 \\
\Rightarrow \lambda^2 - 9x^4 = 0 &\Rightarrow \lambda = \pm 3x^2 \\
\frac{dy}{dx} = 3x^2 \Rightarrow y &= x^3 + \text{const} \Rightarrow y - x^3 = \eta. \\
\frac{dy}{dx} = -3x^2 \Rightarrow y &= -x^3 + \text{const} \Rightarrow y + x^3 = \nu.
\end{aligned}$$

$$u_x = u_\eta(-3x^2) + u_\nu(3x^2), \quad u_y = u_\eta + u_\nu$$

$$\begin{aligned}
\Rightarrow 3x^2(u_\nu - u_\eta) - 3x^2(v_\eta + v_\nu) &= 12x^2 y \\
3x^2(v_\nu - v_\eta) - 3x^2(u_\eta + u_\nu) &= -12x^5 \\
u_\nu - u_\eta - v_\eta - v_\nu &= 4y \\
v_\nu - v_\eta - u_\eta - u_\nu &= -4x^3 \\
\text{Add:} \ -2u_\eta - 2v_\eta &= 4\eta \\
\text{Subtract:} \ 2u_\nu - 2v_\nu &= 4\nu \\
\Rightarrow u + v &= f(\nu) - \eta^2, \\
u - v &= g(\eta) + \nu^2.
\end{aligned}$$

$$u(x,0) = 2x^6, \quad v(x,0) = -x^6$$

$$\Rightarrow x^6 = f(x^3) - x^6, \quad 3x^6 = x^6 + g(-x^3)$$

$$\Rightarrow f(x) = 2x^2, \quad g(x) = 2x^2$$

$$\begin{aligned}
\Rightarrow u &= \tfrac{1}{2}[f(\nu) + g(\eta)] + \tfrac{1}{2}(\nu^2 - \eta^2), \\
&= [(y + x^3)^2 + (y - x^3)^2] \\
&\quad + \tfrac{1}{2}[(y + x^3)^2 - (y - x^3)^2] \\
&= 2(y^2 + x^6) + 2yx^3.
\end{aligned}$$

Similarly

$$\begin{aligned}
u &= \tfrac{1}{2}[f(\nu) - g(\eta)] - \tfrac{1}{2}(\nu^2 + \eta^2), \\
&= [(y + x^3)^2 - (y - x^3)^2] \\
&\quad - \tfrac{1}{2}[(y + x^3)^2 + (y - x^3)^2] \\
&= 4yx^3 - (y^2 + x^6).
\end{aligned}$$

5. Associated quadratic

$$
\begin{aligned}
y\lambda^2 + (y-x)\lambda - x =& 0 \\
\Rightarrow \lambda =& \frac{x - y \pm \sqrt{(x-y)^2 + 4xy}}{2y} \\
=& \frac{x - y \pm (x+y)}{2y} = \frac{x}{y} \text{ or } -1. \\
\frac{dy}{dx} =& 1 \Rightarrow x - y = \text{constant} = \eta \\
\frac{dy}{dx} =& -\frac{x}{y} \Rightarrow x^2 + y^2 = \text{constant} = \nu \\
\Rightarrow u_x = u_\eta + 2xu_\nu \quad u_y = -u_\eta + 2yu_\nu & \\
u_{xx} =& u_{\eta\eta} + 4xu_{\eta\nu} + 2u_\nu + 4x^2u_{\nu\nu} \\
u_{xy} =& -u_{\eta\eta} + 2yu_{\eta\nu} - 2xu_{\eta\nu} + 4xyu_{\nu\nu} \\
u_{yy} =& u_{\eta\eta} - 4yu_{\eta\nu} + 4y^2u_{\nu\nu} + 2u_\nu \\
\Rightarrow [y - (y-x) - x]u_{\eta\eta} & \\
+[4x^2y + 4xy(y-x) - 4xy^2]u_{\nu\nu} & \\
+[4xy + 2(y-x)^2 + 4xy]u_{\eta\nu} & \\
+2(y-x)u_\nu + \frac{x-y}{x+y}(u_x + u_y) =& 2(x+y)^2(x^2+y^2) \\
\Rightarrow u_{\eta\nu} =& \nu \\
\Rightarrow u_\eta =& \tfrac{1}{2}\nu^2 + \tilde{f}(\eta) \\
\Rightarrow u =& \tfrac{1}{2}\nu^2\eta + f(\eta) + g(\nu) \\
=& \tfrac{1}{2}(x^2+y^2)^2(x-y) + f(x-y) + g(x^2+y^2).
\end{aligned}
$$

6. (i) Clearly $\Phi(x) = 6x - 2$.

(ii)

$$w = \cos z = \frac{1}{2}\left(e^{ix-y} + e^{-ix+y}\right)$$
$$= \frac{1}{2}\left[e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)\right]$$
$$\Rightarrow u = \cos x \cosh y, \quad v = -\sin x \sinh y.$$

So $x = \frac{1}{2}$ gets transformed to $u = \cos\frac{1}{2}\cosh y$, $v = -\sin\frac{1}{2}\sinh y$, which satisfy

$$\frac{u^2}{\cos^2\frac{1}{2}} - \frac{v^2}{\sin^2\frac{1}{2}} = 1$$

which is the hyperbola H_1. Similarly

$$\frac{u^2}{\cos^2 1} - \frac{v^2}{\sin^2 1} = 1$$

is the hyperbola H_2.

(iii) Since $\Phi(x)$ is harmonic in the region between $x = \frac{1}{2}$ and $x = 1$, $\Phi(u, v) = \Phi(x) = 6x(u, v) - 1$ is harmonic between H_1 and H_2. u, v are related to x by

$$\frac{u^2}{\cos^2 x} - \frac{v^2}{\sin^2 x} = 1.$$

(iv) For $u^2 = \frac{27}{4}$, $v^2 = 2$, we have (writing $\cos^2 x = C$)

$$\frac{27}{4}(1-C) - 2C = C(1-C)$$
$$\Rightarrow 4C^2 - 39C + 27 = 0$$
$$(C-9)(4C-3) = 0 \Rightarrow C = 9 \text{ or } C = \frac{3}{4}.$$

Must take $C = \frac{3}{4} \Rightarrow x = \frac{\pi}{6}$ (since $\frac{1}{2} < \frac{\pi}{6} < 1$), and so $\Phi(u, v) = \pi - 2$.

7.

$$F(f'(x);\omega) = \int_{-\infty}^{\infty} f' e^{-i\omega x} dx$$
$$= \left[f e^{-i\omega t} \right]_{-\infty}^{\infty} + i\omega \int_{-\infty}^{\infty} f e^{-i\omega t} dx$$
$$= i\omega \overline{f}(\omega).$$

If $g(x) = e^{-a|x|}$ then

$$\overline{g}(\omega) = \int_{-\infty}^{0} e^{ax - i\omega x} dx + \int_{0}^{\infty} e^{-ax - i\omega x} dx$$
$$= \left[\frac{1}{a - i\omega} \right]_{-\infty}^{0} + \left[-\frac{1}{a + i\omega} \right]_{0}^{\infty}$$
$$= \frac{1}{a - i\omega} + \frac{1}{a + i\omega} = \frac{2a}{a^2 + \omega^2}.$$

Fourier transform the PDE $\Rightarrow$

$$\frac{\partial}{\partial t}\overline{u} = -\omega^2 c^2 \overline{u}$$
$$\Rightarrow \overline{u} = e^{-\omega^2 c^2 t}\overline{u}(\omega, 0)$$
$$\Rightarrow u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\omega^2 c^2 t + i\omega x} \overline{u}(\omega, 0) d\omega$$
$$\Rightarrow g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \overline{u}(\omega, 0) d\omega$$
$$\Rightarrow \overline{u}(\omega, 0) = \overline{g}(\omega)$$
$$\Rightarrow u(x,t) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-\omega^2 c^2 t + i\omega x}}{a^2 + \omega^2} d\omega$$
$$= F^{-1}\left(e^{-\omega^2 c^2 t} \overline{g}(\omega) \right)$$
$$= \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-a|x-z| - \frac{x^2}{4c^2 t}} dz.$$