MATH323 FURTHER METHODS OF APPLIED MATHEMATICS

JANUARY 2007

Full marks can be obtained for complete answers to FIVE questions. Only the best five answers will be taken into account.

1. Using the method of variation of arbitrary constants, find the solution of the ordinary differential equation

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = \frac{e^{6x}}{1+e^{2x}}$$

which satisfies $y(0) = \frac{1}{2}\ln 2$, $y(\frac{1}{2}\ln 2) = \frac{3}{2}\ln 3$.

[20 marks]

2. The functional $I[y]$ is given by

$$I[y] = \int_a^b F(x, y, y')dx \quad \text{with} \quad y(a) = y_0, \quad y(b) = y_1$$

where y_0 and y_1 are constants.

By using the Euler-Lagrange equation show that if F is not an explicit function of y then the extremal curve when $I[y]$ is stationary is given by

$$\frac{\partial F}{\partial y'} = C$$

where C is a constant.

For the case when

$$F(x, y, y') = x^2 y'(1 + \tfrac{1}{2}y')$$

find the extremal curve for the boundary conditions $y(1) = 0$, $y(2) = -\frac{3}{2}$. Calculate the corresponding extremal value of $I[y]$, and compare it with the value of I for a straight line joining the two endpoints $(1, 0)$ and $(2, -\frac{3}{2})$. What do you conclude about the nature of the extremum?

Comment briefly on how the derivation of the extremal solution for the above functional would be affected if the limits of the integral were $a = -1$ and $b = 2$ instead of $a = 1$ and $b = 2$.

[20 marks]

3. Indicate briefly how you would find the function $y(x)$ satisfying $y(a) = y_0$ and $y(b) = y_1$, such that the functional

$$I[y] = \int_a^b F(x, y, y')dx$$

is stationary subject to the condition that a second functional

$$J[y] = \int_a^b G(x, y, y')dx$$

is equal to a constant.

For the case

$$I[y] = \int_1^2 [x^4(y')^2 - 2x^2y^2]dx$$

and

$$J[y] = \int_1^2 x^2 y dx = 2$$

where $y(1) = -3$ and $y(2) = 2$, find the extremal curve.

[You do not need to evaluate $I[y]$ for the extremal curve.]

[20 marks]

4. The functions $u(x, y)$ and $v(x, y)$ satisfy the simultaneous partial differential equations

$$\begin{aligned} u_x - 3x^2 v_y &= 12x^2 y \\ v_x - 3x^2 u_y &= -12x^5, \end{aligned}$$

where $x \neq 0$.

Show that this system of differential equations is hyperbolic with characteristics

$$\begin{aligned} y - x^3 &= \eta = \text{constant} \\ y + x^3 &= \nu = \text{constant.} \end{aligned}$$

Hence show by changing variables from (x, y) to (η, ν) that u and v satisfy

$$u_\eta + v_\eta = -2\,\eta, \quad u_\nu - v_\nu = 2\nu.$$

Show that the solution for $u(x, y)$ such that $u(x, 0) = 2x^6$ and $v(x, 0) = -x^6$ is

$$u(x, y) = 2(y^2 + yx^3 + x^6),$$

and give the corresponding solution for $v(x, y)$.

[20 marks]

5. Show that for $x + y \neq 0$, the partial differential equation satisfied by $u(x, y)$,

$$yu_{xx} + (y - x)u_{xy} - xu_{yy} + \frac{x - y}{x + y}(u_x + u_y) = 2(x + y)^2(x^2 + y^2)$$

is hyperbolic with characteristics

$$\eta = x - y = \text{constant} \quad \text{and} \quad \nu = x^2 + y^2 = \text{constant}.$$

Show that the canonical form of the partial differential equation is

$$u_{\eta\nu} = \nu.$$

Find the general solution for $u(x, y)$ when $x + y \neq 0$.

[20 marks]

6. (i) A function $\Phi(x)$ is harmonic in the z-plane, where $z = x + iy$ and x and y are real. In the region $\frac{1}{2} \le x \le 1$, determine $\Phi(x)$ such that $\Phi(\frac{1}{2}) = 1$ and $\Phi(1) = 4$.

[2 marks]

(ii) A conformal transformation is defined by

$$w = \cos z = \tfrac{1}{2}(e^{iz} + e^{-iz}), \text{ for } z \neq n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

where $w = u + iv$, and u and v are real. Show that

$$u(x, y) = \cos x \cosh y$$

and give the corresponding expression for $v(x, y)$. Hence show that the lines $x = \frac{1}{2}$ and $x = 1$ are mapped onto two hyperbolae in the w-plane denoted respectively by H_1 and H_2.

[You do not need to prove the fact that the region $\frac{1}{2} < x < 1$ is mapped into the region R between the branches of the two hyperbolae H_1 and H_2 in the right-hand half of the w-plane, i.e. $u > 0$.]

[6 marks]

(iii) Clearly giving your reasoning, show that the solution $\Phi(u, v)$ to Laplace's equation in this region R, and satisfying $\Phi = 1$ on H_1 and $\Phi = 4$ on H_2, is given by

$$\Phi(u, v) = 6x(u, v) - 2$$

where $x(u, v)$ is the solution to

$$\frac{u^2}{\cos^2 x} - \frac{v^2}{\sin^2 x} = 1. \qquad (*)$$

[5 marks]

(iv) Writing $\cos^2 x = C$ in $(*)$ above, show that if $(u, v) = \left(\frac{3\sqrt{3}}{2}, \sqrt{2}\right)$, then C satisfies

$$4C^2 - 39C + 27 = 0.$$

Hence show that the value of x corresponding to these values of u and v, and satisfying $\frac{1}{2} \le x \le 1$, is $x = \frac{\pi}{6}$. Evaluate $\Phi(u, v)$ for these values of u and v.

[7 marks]

7. The Fourier transform of a function $f(x)$ suitably defined on $-\infty < x < \infty$ is

$$F(f(x);\omega) = \overline{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx.$$

Show that if $f(x) \to 0$ as $x \to \pm\infty$ then

$$F(f'(x);\omega) = i\omega\overline{f}(\omega).$$

Show that the Fourier transform of the function $g(x) = e^{-a|x|}$ (where a is a constant, $a > 0$) is

$$\overline{g}(\omega) = \frac{2a}{a^2+\omega^2}.$$

The function $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

for $-\infty < x < \infty$ and $t \geq 0$. In addition u satisfies the conditions

$$u(x,0) = g(x) = e^{-a|x|}, \quad u, u_x \to 0 \text{ as } x \to \pm\infty.$$

By using a Fourier transform show that $u(x,t)$ is given by

$$u(x,t) = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{e^{-c^2\omega^2 t + i\omega x}}{a^2+\omega^2}\,d\omega.$$

Hence show that

$$u(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-a|x-z|-\frac{z^2}{4c^2 t}}\,dz.$$

[You may use without proof the results

$$F(e^{-b^2x^2};\omega) = \frac{\sqrt{\pi}}{b}e^{-\frac{\omega^2}{4b^2}}$$

$$\text{and } F^{-1}(\overline{f}(\omega)\overline{g}(\omega)) = \int_{-\infty}^{\infty} f(x-z)g(z)dz,$$

where b is a constant, $b > 0$.]