2MA63 JAN 1999

Instructions to candidates

Full marks can be obtained for complete answers to FIVE questions. Only the best five answers will be taken into account.

1. Write down the differential equation satisfied by the function $y(x)$ for which the functional

$$I[y] = \int_a^b F(x, y, y')\, dx \qquad y(a) = y_0, \quad y(b) = y_1$$

is stationary.

Show that the function $y(x)$ which extremises the functional

$$I[y] = \int_1^2 (x^2 y'^2 + 6y^2)\, dx \qquad y(1) = 1, \quad y(2) = 4$$

must satisfy the equation

$$x^2 \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - 6y = 0.$$

By seeking a solution of the form $y = Ax^n$ where A and n are constants, determine the extremal curve $y(x)$ and evaluate the corresponding extreme value of I. Compare this result to the value of I obtained for a straight line joining the two end points. What do you think is the nature of the extremum?

2. Indicate briefly how you would find the function $y(x)$, satisfying $y(a) = y_0, \quad y(b) = y_1$, such that the functional

$$I[y] = \int_a^b F(x, y, y')\, dx$$

is stationary, subject to the condition that a second functional

$$J[y] = \int_a^b G(x, y, y')\, dx$$

is equal to a constant.

For the case

$$I[y] = \int_0^1 (4xy + \tfrac{1}{2} y'^2)\, dx$$

and

$$J[y] = \int_0^1 y\, dx = 1$$

where $y(0) = 0, \quad y(1) = 1$, find the extremal curve. (You need not evaluate the corresponding extreme value of I).

3. The functions $u(x, y)$ and $v(x, y)$ satisfy the simultaneous partial differential equations

$$\begin{aligned} \frac{\partial u}{\partial x} - x\frac{\partial v}{\partial y} &= xy \\ -x\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} &= -xy, \quad x \neq 0, y \neq 0. \end{aligned}$$

Show that this system of differential equations is hyperbolic, with characteristics which may be chosen as follows

$$\begin{aligned} \alpha &= y - \tfrac{1}{2}x^2 \\ \beta &= y + \tfrac{1}{2}x^2. \end{aligned}$$

Hence show by changing variables $(x, y) \to (\alpha, \beta)$ that u and v satisfy the equations:

$$\begin{aligned} \frac{\partial u}{\partial \alpha} + \frac{\partial v}{\partial \alpha} &= 0 \\ \frac{\partial u}{\partial \beta} - \frac{\partial v}{\partial \beta} &= \tfrac{1}{2}(\alpha + \beta). \end{aligned}$$

Find the general solution for u and v in terms of x and y.

4. The function $z(x, y)$ satisfies the partial differential equation

$$y^2\frac{\partial^2 z}{\partial x^2} + x^2\frac{\partial^2 z}{\partial y^2} = F(x, y, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}), \quad x \neq 0, y \neq 0. \tag{1}$$

Show that this equation is elliptic and that it can be reduced to canonical form by the change of variables

$$\nu = y^2 \quad \text{and} \quad \eta = x^2.$$

Hence show that for the case

$$F = \frac{y^2}{x}\frac{\partial z}{\partial x} + \frac{x^2}{y}\frac{\partial z}{\partial y}, \tag{2}$$

z satisfies Laplace's equation in terms of the new variables (ν, η).

By considering the function $\ln Z$, where $Z = \nu + i\eta$, or otherwise, show that a solution to Eq. (1) with F as given in Eq. (2) is

$$z = \ln(x^4 + y^4).$$

5.

(i) Show that the most general solution of Laplace's equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

that is independent of y is given by

$$\Phi = A + Bx$$

where A and B are constants. Determine A, B given that $\Phi = 2$ when $x = -a$ and $\Phi = 6$ when $x = a$.

(ii) Given a function of a complex variable $f(z)$ that is analytic in a region $\mathcal{R}$, state the condition that $f(z)$ must satisfy so that the transformation

$$w = u + iv = f(z)$$

is conformal at all points in $\mathcal{R}$.

Show that the transformation

$$w = e^{\frac{\pi z}{a}}, \tag{1}$$

where a is a positive real constant, is conformal for all z.

Show that the transformation of Eq. (1) maps the region bounded by the square in the (x, y) plane with vertices at $(a, a), (a, -a), (-a, a)$ and $(-a, -a)$ into the region between two concentric circles in the u, v plane of radii e^{π} and $e^{-\pi}$. Indicate in a diagram where each vertex of the square is mapped to in the (u, v) plane.

(iii) Using the results of (i) and (ii), or otherwise, determine the potential $\Phi(u, v)$ which satisfies Laplace's equation in the region between the two concentric circles

$$u^2 + v^2 = e^{2\pi}$$
$$u^2 + v^2 = e^{-2\pi}$$

in the (u, v) plane, with values $\Phi = 2$ and $\Phi = 6$ on the inner and outer circles respectively.

6. The Fourier cosine transform of a function $f(x)$ is defined on the interval $0 < x < \infty$ as follows:

$$F_c\{f(x);\omega\} = \overline{f}_c(\omega) = \int_0^\infty f(x)\cos(\omega x)\,dx,$$

and the corresponding inverse transform is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \overline{f}_c(\omega)\cos(\omega x)\,d\omega.$$

Show that if

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = 0$$

then

$$F_c\{f''(x);\omega\} = -\omega^2\overline{f}_c(\omega) - f'(0).$$

The function $U(x,t)$ satisfies the partial differential equation

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}$$

for $0 < x < \infty$ and $0 < t < \infty$, where $k > 0$. U also satisfies the following conditions:

$$U(x,0) = g(x)$$
$$\frac{\partial U}{\partial x}(0,t) = 0.$$

Using a Fourier cosine transform or otherwise, show that

$$U(x,t) = \frac{2}{\pi}\int_0^\infty e^{-k\omega^2 t}\overline{g}_c(\omega)\cos(\omega x)\,d\omega.$$

Hence show that if $g(x) = e^{-ax^2}$, where $a > 0$, then

$$U(x,t) = \frac{e^{-\frac{ax^2}{1+4akt}}}{\sqrt{1+4akt}}.$$

[You may use without derivation the fact that the Fourier cosine transformation of $g(x) = e^{-ax^2}$ is given by $\overline{g}_c(\omega) = \frac{1}{2}\sqrt{\frac{\pi}{a}}e^{-\frac{\omega^2}{4a}}$.]

7. Describe the Method of Variation of Arbitrary Constants for the solution of ordinary linear differential equations.

Verify that the general solution of the differential equation

$$(x \sin x + \cos x)\frac{d^2y}{dx^2} - x\cos x\frac{dy}{dx} + y\cos x = 0$$

is given by

$$y = C_1 x + C_2 \cos x$$

where C_1 and C_2 are arbitrary constants.

Using the Method of Variation of Arbitrary Constants, show that the solution to the equation

$$(x \sin x + \cos x)\frac{d^2y}{dx^2} - x\cos x\frac{dy}{dx} + y\cos x = f(x)$$

may be written in the form

$$y = L_1(x)x + L_2(x)\cos x$$

where

$$\begin{aligned}\frac{dL_1}{dx} &= \frac{f(x)\cos x}{(x\sin x + \cos x)^2}\\ \frac{dL_2}{dx} &= -\frac{xf(x)}{(x\sin x + \cos x)^2}.\end{aligned}$$

Hence find the general solution for the case

$$f(x) = (x\sin x + \cos x)^2.$$