

**Instructions to candidates**

Full marks can be obtained for complete answers to FIVE questions. Only the best five answers will be taken into account.

1. Write down the differential equation satisfied by the function  $y(x)$  for which the functional

$$I[y] = \int_a^b F(x, y, y') dx \quad y(a) = y_0, \quad y(b) = y_1$$

is stationary.

Show that if  $F$  is not an explicit function of  $x$  then the extremal curve satisfies the equation

$$F - y' \frac{\partial F}{\partial y'} = C$$

where  $C$  is a constant.

For the case

$$I[y] = \int_0^1 y^2(1 - y'^2) dx \quad y(0) = 0, \quad y(1) = 1$$

show that the extremal curve satisfies the equation

$$\frac{dy}{dx} = \frac{\sqrt{c^2 - y^2}}{y}$$

where  $c$  is a constant. Hence find the extremal curve. Calculate  $I[y]$  both for the extremal curve and for the straight line joining the two end points. What can you conclude about the nature of the extremum?

**2.** A dynamical system with one degree of freedom  $q(t)$  is described by the Lagrangian  $L(q, \dot{q}, t)$ , where  $\dot{q} \equiv \frac{dq}{dt}$ . Write down Lagrange's equation.

The Hamiltonian  $H$  is defined as follows:

$$H(q, p, t) = p\dot{q} - L$$

where  $p = \frac{\partial L}{\partial \dot{q}}$ . Show that, given Lagrange's equation, it follows that Hamilton's equations:

$$\frac{\partial H}{\partial q} = -\dot{p} \quad \text{and} \quad \frac{\partial H}{\partial p} = \dot{q}$$

are also true.

A particle of mass  $m$  in motion on the  $x$ -axis in a potential  $V(x)$  is described by the Hamiltonian:

$$H(x, p, t) = \frac{p^2}{2m} e^{-\alpha t} + V(x)$$

where  $\alpha$  is a positive constant. Write down Hamilton's equations for the system. Given initial conditions  $x(0) = 0$  and  $p(0) = p_0$ , find  $x(t)$  for the two cases

(i)

$$V = V_0, \quad \text{where } V_0 \text{ is constant}$$

(ii)

$$V = kx, \quad \text{where } k \text{ is a constant.}$$

For both cases, describe what happens to the particle as  $t \rightarrow \infty$ .

**3.** The functions  $u(x, y)$  and  $v(x, y)$  satisfy the simultaneous partial differential equations

$$2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + 4\frac{\partial v}{\partial y} = 0$$

$$3\frac{\partial u}{\partial x} + 3\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Show that this system of differential equations is hyperbolic, with characteristics:

$$x + y = \beta = \text{constant}$$

$$x + 6y = \alpha = \text{constant}$$

Hence show that the Riemann invariants of the system are  $u + 3v = \text{constant}$  and  $3u + 4v = \text{constant}$ .

Find solutions  $u(x, y)$  and  $v(x, y)$  such that  $u(x, 0) = v(x, 0) = x^2$ , and verify that they satisfy the original equations.

4. Show that the partial differential equation

$$x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} = F(x, y, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) \quad (1)$$

is hyperbolic, with characteristics

$$\nu = xy \quad \text{and} \quad \eta = y/x.$$

Hence show that Equation (1) may be written in canonical form as:

$$4\eta\nu \frac{\partial^2 z}{\partial \eta \partial \nu} - 2\eta \frac{\partial z}{\partial \eta} = -F.$$

Find the general solution of Equation (1) in terms of  $x$  and  $y$  for the case

$$F = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} + 4y^2.$$

5. (i) Show that the conformal transformation

$$w = i \left( \frac{1 - z}{1 + z} \right)$$

maps the interior of the unit circle  $|z| = 1$  into the upper half  $w$ -plane,  $\text{Im } w > 0$ . Indicate in a diagram how the two semicircles  $|z| = 1, \text{Im } z > 0$  and  $|z| = 1, \text{Im } z < 0$  are mapped into the  $w$ -plane.

(ii) Show that  $\Phi = 1 - \frac{1}{\pi} \phi$ , where  $w = u + iv = \rho e^{i\phi}$ , is a function harmonic in the upper-half  $w$ -plane, taking the values  $\Phi = 0$  for  $u < 0, v = 0$  and  $\Phi = 1$  for  $u > 0, v = 0$ .

(iii) Using (i) and (ii), construct a function harmonic inside the unit circle  $|z| = 1$ , taking prescribed values  $F(\theta)$  on its circumference as follows:

$$\begin{aligned} F(\theta) &= 1 \quad \text{for} \quad 0 < \theta < \pi \\ &= 0 \quad \text{for} \quad \pi < \theta < 2\pi. \end{aligned}$$

**6.** Given that the Fourier Transform of a function  $f(x)$  suitably defined on the interval  $-\infty < x < \infty$  is

$$F\{f(x); \omega\} = \overline{f}(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx,$$

show that

$$F\{e^{-a^2 x^2}; \omega\} = \frac{\sqrt{\pi}}{a} e^{-\frac{\omega^2}{4a^2}},$$

where  $a$  is a positive constant.

The function  $u(x, t)$  satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2},$$

for  $t \geq 0$  and with  $c$  constant, and boundary conditions as follows:

$$u(x, 0) = g(x), \quad u, u_x \rightarrow 0 \text{ as } x \rightarrow \pm\infty.$$

By using a Fourier Transform, show that  $u(x, t)$  is given by:

$$u(x, t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} g(x + 2cz\sqrt{t}) e^{-z^2} dz.$$

Hence find  $u(x, t)$  for the case  $g(x) = A\delta(x)$ , where  $A$  is a constant, and  $\delta(x)$  is the Dirac  $\delta$ -function.

**7.** Describe the Method of Variation of Arbitrary Constants for the solution of ordinary linear differential equations.

Using the method, or otherwise, find the general solution of the differential equation

$$\frac{d^2 y}{dx^2} - 4 \frac{dy}{dx} + 3y = \frac{1}{1 + e^{-x}}.$$