PAPER CODE NO.
MATH323

THE UNIVERSITY
of LIVERPOOL

JANUARY 2006 EXAMINATIONS

Bachelor of Arts : Year 3
Bachelor of Science : Year 2
Bachelor of Science : Year 3
Master of Mathematics : Year 3
Master of Mathematics : Year 4
Master of Physics : Year 3
Master of Science : Year 1
No qualification aimed for : Year 1

FURTHER METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks will be awarded for complete answers to five questions. Only the best five answers will be taken into account.

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1. Using the method of variation of arbitrary constants, find the general solution of the ordinary (Euler-Cauchy) differential equation

$$4x^2\frac{d^2y}{dx^2} - 8x\frac{dy}{dx} + 9y = 4x^{3/2}\ln x \ .$$

[You may use, without proof, the result

$$\int \frac{\ln^n x}{x}\, dx = \frac{1}{(n+1)}\ln^{n+1} x + \text{constant} \ .\Bigg]$$

[20 marks]

2. Write down the differential equation satisfied by $y(x)$ for which the functional

$$I[y] = \int_a^b F(x,y,y')\, dx \quad \text{with} \quad y(a) = y_0 \ , \quad y(b) = y_1$$

is stationary where y_0 and y_1 are constants.
Find the function $y(x)$ which extremises the functional

$$I[y] = \int_{-1}^{1} \left[(y')^2 + y^2\right] dx$$

with $y(1) = 2$ and $y(-1) = -2$.
By denoting this explicit solution by y_{ext} show that

$$I[y_{\text{ext}}] = 8\coth 1 \ .$$

Compare this result to the value of I obtained for a straight line joining the endpoints. What do you conclude about the nature of the extremum?

[20 marks]

3. Indicate briefly how you would find the function $y(x)$ satisfying $y(a) = y_0$ and $y(b) = y_1$, such that the functional

$$I[y] \;=\; \int_a^b F(x, y, y')\, dx$$

is stationary subject to the condition that a second functional

$$J[y] \;=\; \int_a^b G(x, y, y')\, dx$$

is equal to a constant.
For the case

$$I[y] \;=\; \int_1^2 \left[x^4 \left(y'\right)^2 \;+\; 4x^2y^2\right]\, dx$$

and

$$J[y] \;=\; \int_1^2 x^4 y\; dx \;=\; 1$$

where $y(1) = 1$ and $y(2) = 0$, find the extremal curve.
[You do not need to evaluate $I[y]$ for the extremal curve.]

[20 marks]

4. The functions $u(x, y)$ and $v(x, y)$ satisfy the simultaneous partial differential equations

$$\begin{aligned} u_x \;-\; 2u_y \;+\; v_y \;&=\; 0 \\ -\; 3u_x \;+\; v_x \;+\; 2v_y \;&=\; 0\;. \end{aligned}$$

Show that this system of differential equations is hyperbolic with characteristics which may be chosen as

$$\begin{aligned} x \;+\; y \;&=\; \eta \;=\; \text{constant} \\ y \;-\; 4x \;&=\; \nu \;=\; \text{constant}\;. \end{aligned}$$

Hence show that the Riemann invariants of the system are

$$u \;-\; v \;=\; \text{constant} \quad \text{and} \quad 6u \;-\; v \;=\; \text{constant}\;.$$

Find the solution for $u(x, y)$ and $v(x, y)$ in terms of x and y such that $u(x, 0) = \frac{1}{6}x^3$ and $v(x, 0) = x$.

[20 marks]

5. Show that for $x^2 \neq 2$, the partial differential equation satisfied by $u(x,y)$,

$$(x^2-1)u_{xx} + x^2 u_{xy} + u_{yy} - \frac{2x}{(x^2-2)}(u_x+u_y) = \frac{(x-y)(x^2-2)^2}{(x^2-1)}$$

is hyperbolic with characteristics

$$\eta = x - y = \text{constant} \quad \text{and} \quad \nu = y - \frac{1}{2}\ln\left|\frac{(x-1)}{(x+1)}\right| = \text{constant} .$$

Show that the canonical form of the partial differential equation is

$$u_{\eta\nu} = \eta .$$

Find the general solution for $u(x,y)$ when $x^2 \neq 2$.

[20 marks]

6. (a) Show that the most general solution of Laplace's equation

$$\Phi_{xx} + \Phi_{yy} = 0$$

which is independent of y is given by $\Phi = Px + Q$ where P and Q are constants.
Determine P and Q given that $\Phi = 1$ on $x = 0$ and $\Phi = -1$ on $x = a$ where a is a constant.

[3 marks]

(b) The rectangle $OABC$ in the z-plane has corners at $O(0,0)$, $A(a,0)$, $B(a,b)$ and $C(0,b)$ where a and b are constants and $a > b$. Show that the conformal map, where $z = x + iy$ and $w = u + iv$,

$$w = \frac{1}{z}$$

maps the line AB into the arc of a circle of radius $1/(2a)$ with centre at $(1/(2a),0)$ beginning and ending at the points $(1/a,0)$ and $(\frac{a}{(a^2+b^2)}, -\frac{b}{(a^2+b^2)})$ in the w-plane.
Determine where the remaining edges of the rectangle are mapped to in the w-plane under $w = 1/z$. Sketch the boundaries of both regions and determine the region the interior of the rectangle is mapped to in the w-plane.

[13 marks]

(c) For the harmonic function $\Phi(x,y)$ satisfying the boundary conditions specified in part (a), determine the corresponding harmonic function in the w-plane, carefully stating your reasoning.

[4 marks]

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7. The Fourier transform of a function $f(x)$ suitably defined on $-\infty < x < \infty$ and its inverse are given respectively by

$$F\left(f(x);\omega\right) = \bar{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$$

$$F^{-1}(\bar{f}(\omega);x) = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(\omega)e^{i\omega x}\,d\omega\ .$$

Show that the Fourier transform of the nth derivative of $f(x)$, denoted by $f^{(n)}(x)$, satisfies

$$F\left(f^{(n)}(x);\omega\right) = (i\omega)^n\bar{f}(\omega)$$

where $f^{(r)}(x) \to 0$ as $x \to \pm\infty$ with $0 \leq r \leq n-1$.
Show that the Fourier transform of the function $f(x) = e^{-a|x|}$ is

$$\bar{f}(\omega) = \frac{2a}{(a^2+\omega^2)}$$

where a is constant and $a > 0$.
The function $\phi(x,y)$ satisfies Laplace's equation for all x and for $y > 0$ subject to the boundary conditions

$$\phi(x,0) = Ae^{-a|x|} \quad \text{and} \quad \phi(x,y) \to 0 \quad \text{as} \quad y \to \infty$$

where A is a constant. Show that

$$\phi(x,y) = \frac{aA}{\pi}\int_{-\infty}^{\infty} d\omega\,\frac{e^{-|\omega|y}e^{i\omega x}}{(a^2+\omega^2)}\ .$$

By assuming that

$$F\left(\frac{b}{\pi(x^2+b^2)};\omega\right) = e^{-|\omega|b}$$

for arbitrary b, use the convolution theorem to deduce

$$\phi(x,y) = \frac{Ay}{\pi}\int_{-\infty}^{\infty} du\,\frac{e^{-a|u|}}{[(x-u)^2+y^2]}\ .$$

[20 marks]