PAPER CODE NO.
MATH323

THE UNIVERSITY
of LIVERPOOL

JANUARY 2005 EXAMINATIONS

Bachelor of Arts : Year 3
Bachelor of Science : Year 3
Master of Mathematics : Year 3
Master of Science : Year 1

FURTHER METHODS OF APPLIED MATHEMATICS

TIME ALLOWED : Two Hours and a Half

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for complete answers to FIVE questions. Only the best five answers will be taken into account.

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1. How many independent solutions are there to an nth order linear ordinary homogeneous differential equation?
Using the method of variation of arbitrary constants, find the general solution of the third order linear ordinary differential equation

$$\frac{d^3y}{dx^3} + \frac{dy}{dx} = \frac{\cos(2x)}{\sin x} .$$

[**20 marks**]

2. The functional $I[y]$ is given by

$$I[y] = \int_a^b F(x,y,y')\,dx \quad \text{with} \quad y(a) = y_0 , \quad y(b) = y_1$$

where y_0 and y_1 are constants.
By using the Euler-Lagrange equation show that if F is not an explicit function of y then the extremal curve when $I[y]$ is stationary is given by

$$\frac{\partial F}{\partial y'} = C$$

where C is a constant.
For the case when

$$F(x,y,y') = y' + (x^2+4)(y')^2$$

show that the extremal curve satisfies

$$\frac{dy}{dx} = \frac{A}{(x^2+4)}$$

where A is a constant. Hence find the extremal curve when $y(0) = 2$ and $y(2) = 3$.
Calculate the value of $I[y]$ for both the extremal curve and the straight line joining the two end points. What do you think the nature of the extremum is?

[**20 marks**]

3. Indicate briefly how you would find the function $y(x)$ satisfying $y(a) = y_0$ and $y(b) = y_1$, such that the functional

$$I[y] \;=\; \int_a^b F(x, y, y')\, dx$$

is stationary subject to the condition that a second functional

$$J[y] \;=\; \int_a^b G(x, y, y')\, dx$$

is equal to a constant.
For the case

$$I[y] \;=\; \int_1^2 \left(2y^2 \;+\; x^2 (y')^2\right)\, dx$$

and

$$J[y] \;=\; \int_1^2 y\, dx \;=\; 1$$

where $y(1) = 2$ and $y(2) = \frac{3}{2}$, find the extremal curve.
[You do not need to evaluate the corresponding value of $I[y]$.]

[20 marks]

4. The functions $u(x, y)$ and $v(x, y)$ satisfy the simultaneous partial differential equations

$$\begin{aligned} y u_x \;-\; v_y \;&=\; x y^3 \\ y v_x \;-\; u_y \;&=\; -\, x y^3 \end{aligned}$$

where $y \neq 0$.
Show that this system of differential equations is hyperbolic with characteristics which may be chosen as

$$\begin{aligned} x \;-\; \tfrac{1}{2} y^2 \;&=\; \eta \;=\; \text{constant} \\ x \;+\; \tfrac{1}{2} y^2 \;&=\; \nu \;=\; \text{constant}\;. \end{aligned}$$

Hence show by changing variables (x, y) to (η, ν) that u and v satisfy

$$u_\eta \;+\; v_\eta \;=\; 0 \quad \text{and} \quad u_\nu \;-\; v_\nu \;=\; \tfrac{1}{2}(\nu^2 - \eta^2)\;.$$

Find the general solution for $u(x, y)$ and $v(x, y)$ in terms of x and y.

[20 marks]

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5. The function $u(x,y)$ satisfies the partial differential equation

$$u_{xx} \; - \; 4xu_{xy} + 4x^2u_{yy} \; = \; F(x,y,u_x,u_y) \; .$$

Show that this equation is parabolic with characteristics

$$\eta \; = \; y \; + \; x^2 \; = \; \text{constant} \qquad \text{and} \qquad \nu \; = \; \nu(x,y) \; = \; \text{constant}$$

where ν is such that $\nu_x \neq 2x\nu_y$.
Choosing $\nu = x$, reduce the original partial differential equation to canonical form.
Find the general solution when

$$F \; = \; u_x \; - \; 2(x-1)u_y \; .$$

[20 marks]

6. (a) A function $\Phi(x)$ is harmonic in the z-plane, where $z = x + iy$. In the region $1 \leq x \leq 2$, determine $\Phi(x)$ such that $\Phi(1) = 1$ and $\Phi(2) = 4$.

[3 marks]

(b) Show that the conformal transformation

$$w \; = \; \cosh z$$

where $z = x + iy$ and $w = u + iv$ transforms the lines $x = 1$ and $x = 2$ into two ellipses in the w-plane denoted respectively by E_1 and E_2.

[5 marks]

(c) By clearly giving your reasoning, show that the solution to Laplace's equation $\Phi(u,v)$ in the region between the ellipses E_1 and E_2 where $\Phi = 1$ on E_1 and $\Phi = 4$ on E_2, is given by

$$\Phi(u,v) \; = \; 3x(u,v) \; - \; 2$$

where $x(u,v)$ is the solution to

$$\frac{u^2}{\cosh^2 x} \; + \; \frac{v^2}{\sinh^2 x} \; = \; 1 \; .$$

[5 marks]

(d) Evaluate $\Phi(u,v)$ at $(u,v) = \left(\frac{17\sqrt{3}}{16}, \frac{15}{16}\right)$.

[7 marks]

7. The Fourier transform of a function $f(x)$ suitably defined on $-\infty < x < \infty$ is

$$F\,(f(x);\omega) \;=\; \bar{f}(\omega) \;=\; \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx \;.$$

Show that if $f(x) \to 0$ as $x \to \pm\infty$ then

$$F(f'(x);\omega) \;=\; i\omega\bar{f}(\omega) \;.$$

The function $g(x)$ is defined as

$$g(x) \;=\; \begin{cases} 1 & \text{if } -1 \le x \le 1 \\ 0 & \text{if } |x| > 1 \;. \end{cases}$$

Show that

$$\bar{g}(\omega) \;=\; \frac{2\sin\omega}{\omega} \;.$$

The function $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} \;=\; 4\frac{\partial^2 u}{\partial x^2}$$

for $-\infty < x < \infty$ and $t \ge 0$. In addition u satisfies the conditions

$$u(x,0) \;=\; g(x) \quad , \quad u, u_x \;\to\; 0 \quad \text{as } x \;\to\; \pm\infty \;.$$

By using a Fourier transform show that $u(x,t)$ is given by

$$u(x,t) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-4\omega^2 t}\,\frac{\sin\omega}{\omega}e^{i\omega x}\,d\omega \;.$$

Hence show that

$$u(x,t) \;=\; \frac{1}{4\sqrt{\pi t}}\int_{-1}^{1} e^{-(x-z)^2/(16t)}\,dz \;.$$

[You may use without proof the results

$$F(e^{-a^2x^2};\omega) \;=\; \frac{\sqrt{\pi}}{a}e^{-\omega^2/(4a^2)}$$

and

$$F^{-1}(\bar{f}(\omega)\bar{g}(\omega)) \;=\; \int_{-\infty}^{\infty} f(x-z)g(z)\,dz \;.\Big]$$

[20 marks]